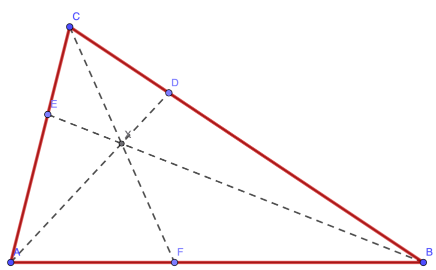
Proof of Ceva's theorem

Repeating the setup for Ceva's theorem, "Let $\triangle ABC$ be a triangle, and let D, E and F be on the lines BC, CA and AB respectively, such that the lines AD, BE and CF are concurrent." Denote the common intersection of these three lines by X. Figure 1 shows two possible configurations for the points A, B, C, D, E, F and X. We will prove that

$$\frac{|AF|}{|FB|} \cdot \frac{|BD|}{|DC|} \cdot \frac{|CE|}{|EA|} = 1\,.$$

Note: we don't expect you to repeat statements unless there's a reason. My reason is that I am going to show two different diagrams for it, emphasizing that the hypotheses do not determine which of the points B, X and E lies between the other two.



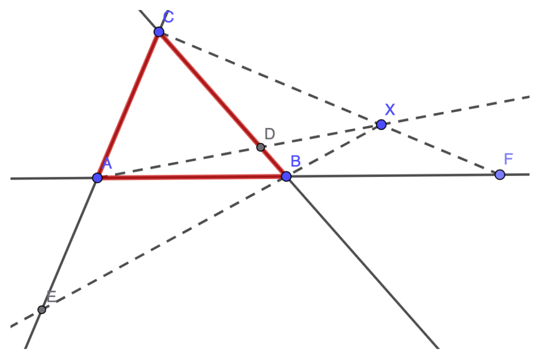


Figure 1: two possible configurations in Ceva's theorem

STEP 1 OF THE PROOF: Draw a line ℓ parallel to AB through the opposite vertex C. Let G denote the intersection of the lines BX and ℓ and let H denote the intersection of the lines AX and ℓ. This is illustrated in Figure 2.

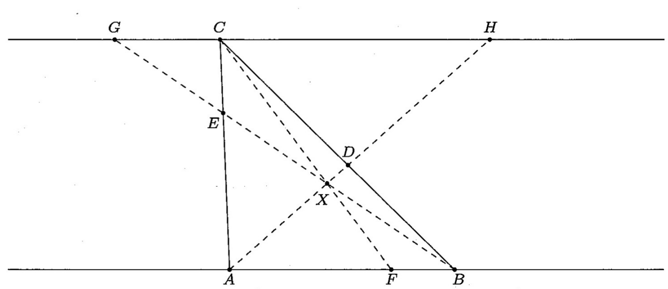


Figure 2: adding a line through C parallel to AB

Step 2 <u>Claim:</u> $\triangle AXF$ is similar to $\triangle HXC$. To prove this, note that $\angle AXF = \angle CXH$ because these are *vertical angles*, both supplementary to the same angle $\angle FXH$; also $\angle XCH = \angle XFA$ because these are alternate interior angles of where the transversal BC cuts the parallel lines ℓ and AB; having established equality of two of the pairs of corresponding angles of the two triangles, it follows that the third pair is equal, hence the triangles are similar. Similarity implies the ratios of corresponding sides are equal:

$$\frac{|\mathbf{AF}|}{|\mathbf{HC}|} = \frac{|\mathbf{FX}|}{|\mathbf{CX}|} = \frac{|XA|}{|XH|}\,. \tag{1}$$

Step 3 <u>Three similar claims:</u> The fact that $\triangle XFB$ is similar to $\triangle XCG$ is proved by the same reasoning, with B in place of A and G in place of H. This yields

$$\frac{|\mathbf{BF}|}{|\mathbf{GC}|} = \frac{|\mathbf{FX}|}{|\mathbf{CX}|} = \frac{|XB|}{|XG|}\,. \tag{2}$$

Lastly, we can find two more pairs of similar triangles, again with completely analogous justifications: $\triangle ABD$ and $\triangle HCD$ are similar, as are $\triangle ABE$ and $\triangle CGE$, yielding the identities

$$\frac{|\mathbf{CD}|}{|\mathbf{BD}|} = \frac{|\mathbf{HC}|}{|\mathbf{AB}|} = \frac{|HD|}{|AD|}\,; \tag{3}$$

$$\frac{|\mathbf{AE}|}{|\mathbf{CE}|} = \frac{|\mathbf{AB}|}{|\mathbf{CG}|} = \frac{|BE|}{|GE|}\,. \tag{4}$$

Last step: some algebra. From the boldface parts of equations (1) and (2) we see that two quanties are both equal to $|FX|/|CX|$, and setting the two quantities equal yields

$$\frac{|AF|}{|FB|} = \frac{|HC|}{|CG|}\,. \tag{5}$$

From (3) we obtain

$$\frac{|BD|}{|DC|} = \frac{|AB|}{|CH|}\,. \tag{6}$$

From (4) we obtain

$$\frac{|CE|}{|EA|} = \frac{|GC|}{|AB|}\,. \tag{7}$$

Multiplying together (5),(6) and (7) proves Ceva's theorem.