Definitions for $\mathbb{E}^3$ (my version)

These definitions are based on $\mathbb{R}^3$, that is, it is assumed we know definitions and names of points, lines and planes in $\mathbb{R}^3$, as well as which points lie on which lines and planes in $\mathbb{R}^3$, and which lines of $\mathbb{R}^3$ are contained in which planes.

The set of **points** of $\mathbb{E}^3$ consist of two kinds.

- **ordinary points** are just the points of $\mathbb{R}^3$
- **ideal points** are denoted $P_{[[k]]}$ where k is some line in $\mathbb{R}^3$ and $[[k]]$ denotes the set of all lines parallel to k (this includes k itself). Therefore, $P_{[[k]]} = P_{[[m]]}$ (they are the same ideal point) if and only if $k \parallel m$.

The set of **lines** of $\mathbb{E}^3$ consist of two kinds.

- **ordinary lines** are lines k_* where k is a line of $\mathbb{R}^3$ and k_* is $k \cup P_{[[k]]}$. That is, as a set, k_* contains all the points of k together with exactly one ideal point, $P_{[[k]]}$.
- **ideal lines** are denoted $\ell_{[[\alpha]]}$ where α is some plane in $\mathbb{R}^3$ and $[[\alpha]]$ denotes the set of all planes parallel to α (this includes α itself). Therefore, $\ell_{[[\alpha]]} = \ell_{[[\beta]]}$ if and only if the planes α and β are parallel (including the case that they are equal).

Ideal lines contain no ordinary points, only ideal points. Specifically, $P_{[[k]]} \in \ell_{[[\alpha]]}$ if and only if k is parallel to α, which includes the case $k \subset \alpha$.

The set of **planes** of $\mathbb{E}^3$ consist of two kinds.

- **ordinary planes** are planes α_* where α is a plane in $\mathbb{R}^3$ and $\alpha_* = \alpha \cup \ell_{[[\alpha]]}$. Thus, an ideal point $P_{[[k]]}$ is in α_* if and only if $k \parallel \alpha$.
- **the ideal plane** denoted π_∞ is defined to be the set of all ideal points. Therefore, a line is in the plane π_∞ if and only if it is an ideal line.

Note: I changed the name of the ideal plane from α_∞ to π_∞ because we often like to use α for a particular plane in some context. You can use pretty much any Greek letter in place of α or π with an infinity subscript and I'll know you mean the plane at infinity, but you might as well use whichever of α or π you find more natural.