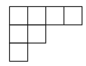
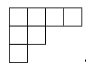
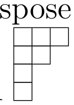
1 Definitions

1.1 Partitions

$\lambda = (\lambda_1, \lambda_2, \ldots)$, $\lambda \vdash n \iff \lambda_1 + \lambda_2 + \cdots = n$.

Young diagram: e.g. $\lambda = (4, 2, 1) \to$ 
Conjugate partition: λ' - transpose of the Young diagram of λ,

e.g. if $\lambda = (4, 2, 1) \to$ , then  $\to \lambda' = (3, 2, 1, 1)$
Dominance order: $\lambda \leq \mu \vdash n$ iff $\lambda_1 + \cdots + \lambda_i \leq \mu_1 + \cdots \mu_i$ for every i.

1.2 Bases of Λ

- **Monomial:** $m_\lambda = \sum_\alpha x^\alpha$, where $\alpha = (\alpha_1, \alpha_2, \ldots)$ ranges among all different permutations of $\lambda = (\lambda_1, \lambda_2, \ldots)$.
- **Elementary:** $e_n = \sum_{i_1 < i_2 < \cdots < i_n} x_{i_1} x_{i_2} \ldots x_{i_n}$, $e_\lambda = e_{\lambda_1} e_{\lambda_2} \ldots$.
- **Complete Homogeneous:** $h_n = \sum_{i_1 \leq i_2 \leq \cdots \leq i_n} x_{i_1} x_{i_2} \ldots x_{i_n} = \sum_{\lambda \vdash n} m_\lambda$, $h_\lambda = h_{\lambda_1} h_{\lambda_2} \cdots$.
- **Power sum:** $p_n = \sum_i x_i^n$, $p_\lambda = p_{\lambda_1} p_{\lambda_2} \cdots$.

1.3 Maps

Endomorphism $\omega : \Lambda \to \Lambda$, defined by $\omega(e_n) = h_n$.
Scalar product $\langle , \rangle : \Lambda \times \Lambda \to \Lambda$, defined by $\langle m_\lambda, h_\mu \rangle = \delta_{\lambda\mu}$.

2 Theorems

Theorem 1. If $\lambda \vdash n$, then $e_\lambda = \sum_{\mu \vdash n} M_{\lambda\mu} m_\mu$, where $M_{\lambda\mu}$ is the number of $(0,1)-$matrices $A = (a_{ij})$ with $\sum_j a_{ij} = \lambda_1$ and $\sum_i a_{ij} = \mu_j$. Hance M is a symmetric matrix and $M_{\lambda\mu} = 0$ unless $\mu \leq \lambda'$ and $M_{\lambda\lambda'} = 1$.

Theorem 2. If $h_\lambda = \sum N_{\lambda\mu} m_\mu$, then $N_{\lambda\mu}$ is the number of $\mathbb{N}$-matrices A with $\sum_j a_{ij} = \lambda_1$ and $\sum_i a_{ij} = \mu_j$. Hence N is symmetric.

Theorem 3. If $p_\lambda = \sum R_{\lambda\mu} m_\mu$, then $R_{\lambda\mu}$ is the number of ordered set partitions $(B_1, \ldots, B_k)$ of $[1, \ldots, l]$ $(k = l(\mu, l = l(\lambda))$, such that $\mu_j = \sum_{i \in B_j} \lambda_i$, $1 \leq j \leq k$. Hence $R_{\lambda\mu} = 0$ unless $\lambda \leq \mu$ and $R_{\lambda\lambda} = m_1! m_2! \ldots$, where $\lambda = 1^{m_1} 2^{m_2} \ldots$.

Theorem 4. We have that

$$\prod_{i,j} \frac{1}{(1-x_i y_j)} = \sum_\lambda h_\lambda(x) m_\lambda(y), \tag{1}$$

$$\prod_{i,j} (1+x_i y_j) = \sum_\lambda e_\lambda(x) m_\lambda(y), \tag{2}$$

$$\prod_{i,j} \frac{1}{(1-x_i y_j)} = \sum_\lambda z_\lambda^{-1} p_\lambda(x) p_\lambda(y), \tag{3}$$

$$\prod_{i,j} (1+x_i y_j) = \sum_\lambda \epsilon_\lambda z_\lambda^{-1} p_\lambda(x) p_\lambda(y), \tag{4}$$

where $z_\lambda = 1^{m_1} m_1! 2^{m_2} m_2! \cdots$, $\epsilon_\lambda = (-1)^{|\lambda| - l(\lambda)}$.

Theorem 5. The endomorphism ω is an involution. Moreover, $\omega(p_\lambda) = \epsilon_\lambda p_\lambda$.

Theorem 6. The scalar product $\langle , \rangle$ is symmetric, i.e. $\langle f, g \rangle = \langle g, f \rangle$ for all $f, g \in \Lambda$. We also have that $\langle p_\lambda, p_\mu \rangle = z_\lambda \delta_{\lambda\mu}$ and hence is positive definite ($\langle f, f \rangle \geq 0$ and is 0 iff $f = 0$).