Spring Tutorial on Algebraic Combinatorics, Symmetric functions

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Time: TBD after sign-up, based on students' schedule and preferences

Location: TBA

Referrence text: Chapter 7 of *Enumerative Combinatorics, Vol.2, Richard Stanley*

Grading policy: Problem sets (choice of 2 problems from a list of 10), due every 2-3 weeks, then project (4-5 page paper)

Young Tableaux

$$\begin{array}{cccc} 1 & 1 & 2 & 5 \\ 2 & 3 & 3 & \\ 5 & 6 & & \end{array}$$

Partitions

$$\lambda = (\lambda_1, \lambda_2, \ldots), \quad \begin{array}{cccc} \square & \square & \square & \square \\ \square & \square & & \\ \square & & & \end{array}$$

RSK algorithm

$$561423 \leftrightarrow \left(\begin{array}{ccc} 1 & 2 & 3 \\ 4 & 6 & \\ 5 & & \end{array}, \begin{array}{ccc} 1 & 2 & 6 \\ 3 & 4 & \\ 5 & & \end{array} \right)$$

The algebra of symmetric functions, Λ

Bases: monomial $m_\lambda = \sum x_{i_1}^{\lambda_1} \cdots x_{i_k}^{\lambda_k}$,

elementary $e_\lambda = \prod_l \sum_{i_1 < i_2 < \cdots < i_{\lambda_l}} x_{i_1} x_{i_2} \cdots x_{i_{\lambda_l}}$,

homogenous $h_\lambda = \prod_l \sum_{i_1 \leq i_2 \leq \cdots \leq i_{\lambda_l}} x_{i_1} x_{i_2} \cdots x_{i_{\lambda_l}}$,

power sum $p_\lambda = \prod_l \sum x_i^{\lambda_l}$ and ...

Schur functions, s_λ

$\prod_{i,j}(1 + x_i y_j) = \sum_\lambda m_\lambda(x) e_\lambda(y)$

$\prod_{i,j}(1 - x_i y_j)^{-1} = \sum_\lambda s_\lambda(x) s_\lambda(y)$

$\langle m_\lambda, h_\mu \rangle = \delta_{\lambda\mu} = \langle s_\lambda, s_\mu \rangle$, etc...

Representation and Lie theory:

Irrep characters of $S_n = \langle p_w, s_\lambda \rangle$

Irrep characters of $GL_n = s_\lambda$

GL_n: $V_\mu \otimes V_\nu = \oplus V_\lambda^{\langle s_\lambda, s_\mu s_\nu \rangle}$

Permutations,
permutation statistics

$\mathrm{is}(4\underline{\mathbf{25}}3\underline{\mathbf{6}}1) = 3$,

$D(561423) = \{3, 5\}$

$= D\left(\begin{array}{ccc} 1 & 2 & 6 \\ 3 & 4 & \\ 5 & & \end{array} \right)$, etc...

Plane partitions:

$\sum_{\pi \in \mathcal{P}(r,c)} q^{\mathrm{tr}(\pi)} x^{|\pi|} = \prod_{i=1}^{r} \prod_{j=1}^{c} (1 - q x^{i+j-1})^{-1}$

($q = 1, r = 1, c = \infty \rightarrow$ Euler's identity

$\sum_\lambda x^{|\lambda|} = \prod (1 - x^i)^{-1}$)

Algebraic Geometry:

Intersection Theory,

Schubert caclulus,

$H^*(Gr(n,k))$, etc