Math 21b, Section 7, In-class exercise SOLUTIONS
Gramm-Schmidt and QR decomposition

Find the QR decomposition of the matrix

$$M = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

by going through the Gramm-Schmidt orthogonalization of its column vectors $\vec{v_1}, \vec{v_2}, \vec{v_3}$ outlined bellow, i.e. fill in the gaps:

Gramm-Schmidt orthogonalization: We first find the orthonormal basis $\{\vec{u_1}, \vec{u_2}, \vec{u_3}\}$ from the original basis $\{\vec{v_1}, \vec{v_2}, \vec{v_3}\}$:

First of all, the vectors $\{\vec{v_1}, \vec{v_2}, \vec{v_3}\}$ are the columns of M:

$$\vec{v_1} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \vec{v_2} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \vec{v_3} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

Next we find the orthonormal vectors $\vec{u_i}$ one by one. $\vec{u_1}$ is just the unit vector proportional to $\vec{v_1}$, i.e.

$$\vec{u_1} = \frac{1}{\|\vec{v_1}\|}\vec{v_1} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}.$$

Next $\vec{u_2}$ should be perpendicular to $\vec{u_1}$ and in the plane spanned by $\vec{v_1}$ and $\vec{v_2}$, so it will be proportional(parallel) to

$$\vec{w_2} = \vec{v_2}^{\perp} = \vec{v_2} - \text{Proj}_{\vec{u_1}}(v_2) = \vec{v_2} - (\vec{v_2} \cdot \vec{u_1})\vec{u_1} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} - (\frac{1}{\sqrt{2}} \cdot 0 + \frac{1}{\sqrt{2}} \cdot 1 + 0 \cdot 0) \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 0 \end{bmatrix}.$$

So it remaines to make the leght 1 ("normalize"): $\vec{u_2} = \frac{1}{\|\vec{w_2}\|}\vec{w_2} = \frac{1}{1/\sqrt{2}} \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}$. Finally we find $\vec{u_3}$ as being the normalized $\vec{w_3} = \vec{v_3} - \text{Proj}_{span\{\vec{v_1}, \vec{v_2}\}}(\vec{v_3})$, so

$$\vec{w_3} = \vec{v_3} - (\vec{v_3} \cdot \vec{u_1})\vec{u_1} - (\vec{v_3} \cdot \vec{u_2})\vec{u_2} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{\sqrt{2}} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix} - (-\frac{1}{\sqrt{2}}) \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

We get then that $\vec{u_3} = \frac{1}{\|\vec{w_3}\|}\vec{w_3} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

QR decomposition: The matrix Q is just the matrix with column vectors $\vec{u_1}, \vec{u_2}, \vec{u_3}$:

$$Q = \begin{bmatrix} | & | & | \\ \vec{u_1} & \vec{u_2} & \vec{u_3} \\ | & | & | \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The matrix R is the change of basis matrix, but its entries can be simply read off the Gramm-Schmidt procedure above:

$$R = \begin{bmatrix} \|\vec{v_1}\| & \vec{u_1} \cdot \vec{v_2} & \vec{u_1} \cdot \vec{v_3} \\ 0 & \|\vec{w_2}\| & \vec{u_2} \cdot \vec{v_3} \\ 0 & 0 & \|\vec{w_3}\| \end{bmatrix} = \begin{bmatrix} \sqrt{2} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & 0 & 1 \end{bmatrix}.$$