Math 21b, Section 7, Handout
Determinants and their properties

Determinants of upper/lower triangular matrices: There is only one pattern which will not contain a 0 and this the pattern of the entries on the diagonal, which you can see by Laplace's expansion: the only nonzero entry in the first row would be a_{11}, next for the determinant of the minor A_{11} again by Laplace - the only nonzero entry of its first row is a_{22} and so on. So we can say that $\boxed{\text{the determinant of a triangular matrix is the product of the entries on the diagonal, } \det A = a_{11}a_{22}a_{33}\dots a_{nn}}$.

Determinants of block matrices: Block matrices are matrices of the form $M = \begin{bmatrix} A & B \\ 0 & D \end{bmatrix}$ or $M = \begin{bmatrix} A & 0 \\ C & D \end{bmatrix}$ with A and D square, say A is $k \times k$ and D is $l \times l$ and 0 - a (necessarily) $l \times k$ matrix with only 0s. *(Please, refer to page 258 of your textbook for nice diagrams.)* If we consider patterns again, all that have an entry 0 will have product 0, so we need to look at the ones which have no entires in the 0 matrix. This means that if P is a pattern without an entry from the 0 matrix, then: P being a pattern must have entries from every column and row. So P has k entries from the first k columns. None of them should be in matrix 0, so they are all from matrix A. since A has already k rows only, these k entries are a complete pattern from A (one for each row and one for each column). Similalry, P has to have an entry in each of the last l rows, but none of this entries should be from 0, so they are all in D. Finaly we just got that k entries of the pattern P are from A and l from D, but since $k + l = n$, these are all the entries in the pattern. P is therefore a union of a pattern in A, P_A and a pattern in D P_D and because A and D are NW/SE to each other, there are no inversions between them. So we have that $sgn(P)Prod(P) = sgn(P_A)Prod(P_A)sgn(P_D)Prod(P_D)$. Plugging this into the original definition we get that

$$\det(M) = \Sigma_{\{P-pattern\}} \operatorname{sgn}(P) \operatorname{Prod}(P) =$$
$$\Sigma_{\{P \text{ pattern with a } 0\}} 0 + \Sigma_{P=P_A+P_D} \operatorname{sgn}(P_A) \operatorname{Prod}(P_A) \operatorname{sgn}(P_D) \operatorname{Prod}(P_D) =$$
$$(\Sigma_{P_A} \operatorname{sgn}(P_A) \operatorname{Prod}(P_A))(\Sigma_{P_D} \operatorname{sgn}(P_D) \operatorname{Prod}(P_D)) = \det(A)\det(D).$$

To summarize, $\boxed{\det M = \det A \det D}$.

Determinant of the transpose A^T: Notice that if two elements are in an inversion, i.e. NE/SW to each other, then after flipping the matrix along the main diagonal these elements will be again in an inversion. Notice also that the collection of patterns in A and A^T is the same as switching rows and columns does not change the condition of "no two in same row, no two in same column". So the patterns will be the same and signatures will be, so we can see that $\boxed{\det A = \det A^T}$.

Finding the determinant of A through row reduction:
Let B be the matrix obtained from A by one row operation, so if the row operation is:

- swapping two rows, then $\det B = -\det A$.
- adding a multiple of a row to another row then $\det B = \det A$.
- scaling a row by k, then $\det B = k \det A$.

These rules allow us to compute $\det A$ directly from the Gauss-Jordan elimination: Determinant of a product of two matrices: We obtain $\operatorname{rref}(A)$ as a sequence of elementary row operations and let B_i be the matrices we get in this sequence. For example if we obtain $\operatorname{rref}(A)$ from A by the following sequence of operations:

$$A \xrightarrow{\text{swap rows}} B_1 \xrightarrow{\text{scale a row by } k_1} B_2 \xrightarrow{\text{subtract a multiple of row from another}} B_3 \xrightarrow{\text{swap rows}} B_4 \xrightarrow{\text{scale a row by } k_2} B_5 = \operatorname{rref}(A),$$

then as we showed in class and by the items above we will have $\det B_1 = (-1)\det A$, $\det B_2 = k_1 \det B_1$, $\det B_3 = \det B_2$, $\det B_4 = (-1)\det B_3$, $\det B_5 = k_2 \det B_4$ and $\det \operatorname{rref}(A) = \det B_5$. Now we can combine all these equalities to get $\det \operatorname{rref}(A) = \det B_5 = k_2 \det B_4 = k_2(-1)\det B_3 = k_2(-1)\det B_2 = k_2(-1)k_1 \det B_1 = k_2(-1)k_1(-1)\det A$, so

$$\det \operatorname{rref}(A) = (-1)^2 k_1 k_2 \det A, \text{ i.e. } \det A = \frac{(-1)^2}{k_1 k_2} \det \operatorname{rref}(A).$$

Since this procedure readily generalizes we can state it as

If rref A was obtained from A by $\boxed{s \text{ swaps and scaling of rows by } k_1, \ldots, k_t}$ (and no matter what or how many subtractions of a row from another), then $\boxed{\det \operatorname{rref}(A) = (-1)^s k_1 \cdots k_t \det A.}$ Since for an $n \times n$ matrix rref A would be upper triangular with only 1s and 0s on the diagonal, we see that $\det \operatorname{rref}(A) = 1$ if $\operatorname{rref}(A) = I_n$ and 0 otherwise (i.e. A is not invertible). So

$\boxed{\det A = \frac{(-1)^s}{k_1 \cdots k_t} \text{ if } A \text{ is invertible and } \det A = 0 \text{ if and only if } A \text{ is not invertible.}}$

The determinant of the product of two matrices:

Let A and B be two $n \times n$ matrices. Let's do row reduction on an augmented matrix

$$[A|AB] \xrightarrow{\text{row reduction}} [\operatorname{rref}(A)|A^{-1}AB = B].$$

Notice that whatever operation we perform on A, the same goes on AB, so by what we just mentioned above about row reduction, if we've performed s swaps and scaled by $k_1, k_2, \ldots, k_l$, then we will have the corresponding scaling of the determinant:

$$\det(\operatorname{rref}(A)) = (-1)^s k_1 \ldots k_l \det(A), \det(B) = (-1)^s k_1 \ldots k_l \det(AB),$$

If $\det(\operatorname{rref}(A)) = 0$, then A is not invertible, and so is AB, so $\det(AB) = 0 = \det(A)\det(B)$. Otherwise $\det(\operatorname{rref}(A)) = 1$ and so $(-1)^s k_1 \ldots k_l = \frac{1}{\det A}$ and $\det(B) = \frac{1}{\det A} \det(AB)$. Finally $\boxed{\det(AB) = \det(A)\det(B)}$.