DIAGONALIZATION IN THE COMPLEX, solutions **Math 21b**

Diagonalize the matrix

$$L = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

1) The characteristic polynomial is $f_A(\lambda) = -\lambda^3 + 1$
2) The eigenvalues are:

The eigenvalues are the roots of the polynomial $\lambda^3 - 1$. Let z be a root of this polynomial, i.e. $z^3 = 1$. The magnitute (absolute value) of z must then be 1 and so z lies on the unit circle in $\mathbb{C}$, i.e. $z = \cos\alpha + i\sin\alpha$ for some α. De Moivre's formula (page 348 in your textbook) tells us that $(cos\alpha + i\sin\alpha)^n = \cos(n\alpha) + i\sin(n\alpha)$. In our case $1 = z^3 = \cos(3\alpha) + i\sin(3\alpha)$, so $\cos(3\alpha) = 1$ and $\sin(3\alpha) = 0$. This means that $3\alpha = 2\pi k$ for some integer k and so we get the possible solutions for z by substituing $\alpha = 2\pi k/3$ for $k = 0, 1, 2$ (notice how after $k = 2$ the values will start repeating).
So the three roots of $\lambda^3 - 1$ and correspondingly eigenvalues of A are $\cos(0) + i\sin(0) = 1, \cos(2\pi/3) + i\sin(2\pi/3) = -\frac{1}{2} + i\frac{\sqrt{3}}{2}, \cos(2\pi 2/3) + i\sin(2\pi 2/3) = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$.
In general, the equation $z^n = a$ has solutions of the form $a^{1/n}(\cos(2\pi k/n) + i\sin(2\pi k/n))$ for $k = 0, 1, \cdots, n-1$ and the numbers $\boxed{\cos(2\pi k/n) + i\sin(2\pi k/n) \text{ are the } n\text{-th } \textbf{roots of unity}}$ (the vertices of a regular $n-$gon with center at 0 and one vertex at 1.
5) The eigenvectors are...
Let z be an eigenvalue for A, i.e. one of $1, -\frac{1}{2} + i\frac{\sqrt{3}}{,} - \frac{1}{2} - i\frac{\sqrt{3}}{2}$. Since these look pretty horrible, we will do our computations symbolically and only substitute with numbers at the end. We know though that $z^3 = 1$.
In order to find the corresponding eigenvector we find the kernel of $A - zI$, by row reduction:

$$\ker\begin{bmatrix} -z & 0 & 1 \\ 1 & -z & 0 \\ 0 & 1 & -z \end{bmatrix} = \ker\begin{bmatrix} 1 & -z & 0 \\ -z & 0 & 1 \\ 0 & 1 & -z \end{bmatrix} = \ker\begin{bmatrix} 1 & -z & 0 \\ -z + z.1 & 0 + z(-z) & 1 + z.0 \\ 0 & 1 & -z \end{bmatrix} =$$

$$\ker\begin{bmatrix} 1 & -z & 0 \\ 0 & -z^2 & 1 \\ 0 & 1 & -z \end{bmatrix} = \ker\begin{bmatrix} 1 & -z & 0 \\ 0 & 1 & -z \\ 0 & -z^2 & 1 \end{bmatrix} =$$

$$\ker\begin{bmatrix} 1 & -z & 0 \\ 0 & 1 & -z \\ 0 & -z^2 + z^2 1 = 0 & 1 + z^2(-z) = 1 - z^3 = 0 \end{bmatrix} = \ker\begin{bmatrix} 1 & -z + z1 = 0 & z(-z) \\ 0 & 1 & -z \\ 0 & 0 & 0 \end{bmatrix} = span\{\begin{bmatrix} z^2 \\ z \\ 1 \end{bmatrix}\}$$

Now we just plug in the actual values of z to get the eigenvectors:

$$z = 1, \vec{v_1} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, z = \cos(2\pi/3) + i\sin(2\pi/3) = -\frac{1}{2} + i\frac{\sqrt{3}}{2}, \vec{v_2} = \begin{bmatrix} -\frac{1}{2} - i\frac{\sqrt{3}}{2} \\ -\frac{1}{2} + i\frac{\sqrt{3}}{2} \\ 1 \end{bmatrix}, z = -\frac{1}{2} - i\frac{\sqrt{3}}{2}, \vec{v_3} = \begin{bmatrix} -\frac{1}{2} + i\frac{\sqrt{3}}{2} \\ -\frac{1}{2} - i\frac{\sqrt{3}}{2} \\ 1 \end{bmatrix}$$

5)If S is the matrix which contains these eigenvectors as columns in the given order, then $S^{-1}AS = D$ is diagonal. This diagonalization is called the **discrete Fourier transform**.

$$S^{-1}AS = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{2} + i\frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & -\frac{1}{2} - i\frac{\sqrt{3}}{2} \end{bmatrix}$$