Math 21b, Section 7, Handout

On change of basis, an explicit treatment

This handout gives a rather explicit treatment of the problem of finding coordinates in different bases, in the hope of showing the issue from a different perspective.

Let's denote the two bases of the same space V by $\mathcal{W} = \{\vec{w_1}, \vec{w_2}, \vec{w_3}\}$ and $\mathcal{U} = \{\vec{u_1}, \vec{u_2}, \vec{u_3}\}$. Here I chose the dimension of V to be 3 (3 vectors in each basis), but the same reasoning applies to any dimension.

Let $\vec{x}$ be a vector in V, whose coordinates in basis $\mathcal{W}$ are known. In other words(i.e. symbols), we have that $\vec{x} = c_1\vec{w_1} + c_2\vec{w_2} + c_3\vec{w_3}$ for some scalars c_1, c_2, c_3. These scalars are the coordinates of $\vec{x}$ in basis $\mathcal{W}$ and we write this as $[\vec{x}]_{\mathcal{W}} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$. For example, $[\vec{w_2}]_{\mathcal{W}} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

The problem of changing basis consists of finding the coordinates d_1, d_2, d_3 of $\vec{x}$ in the other basis, $\mathcal{U}$ and its solution lies in finding the coordinates of $\vec{w_1}, \vec{w_2}, \vec{w_3}$ in the basis $\mathcal{U}$. To see why, let's assume first that we know these coordinates:

$$[\vec{w_1}]_{\mathcal{U}} = \begin{bmatrix} r_{11} \\ r_{21} \\ r_{31} \end{bmatrix}, [\vec{w_2}]_{\mathcal{U}} = \begin{bmatrix} r_{12} \\ r_{22} \\ r_{32} \end{bmatrix}, [\vec{w_3}]_{\mathcal{U}} = \begin{bmatrix} r_{13} \\ r_{23} \\ r_{33} \end{bmatrix}.$$

In other words we have that

$$\vec{w_1} = r_{11}\vec{u_1} + r_{21}\vec{u_2} + r_{31}\vec{u_3}, \tag{1}$$

$$\vec{w_2} = r_{12}\vec{u_1} + r_{22}\vec{u_2} + r_{32}\vec{u_3}, \tag{2}$$

$$\vec{w_3} = r_{13}\vec{u_1} + r_{23}\vec{u_2} + r_{33}\vec{u_3}. \tag{3}$$

What this means in matrix language is simply that

$$\underbrace{\begin{bmatrix} | & | & | \\ \vec{w_1} & \vec{w_2} & \vec{w_3} \\ | & | & | \end{bmatrix}}_{P} = \underbrace{\begin{bmatrix} | & | & | \\ \vec{u_1} & \vec{u_2} & \vec{u_3} \\ | & | & | \end{bmatrix}}_{S} \underbrace{\begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}}_{R}. \tag{4}$$

In the case of $\mathcal{W} = \{\vec{e_1}, \vec{e_2}, \vec{e_3}\}$ the standard basis in $\mathbb{R}^3$, the above equation is simply $I_3 = SR$, so $R = S^{-1}$.

We can easily find the coordinates of $\vec{x}$ in basis $\mathcal{U}$ by just plugging in the last 3 equations into the formula for $\vec{x}$, i.e. $\vec{x} = c_1\vec{w_1} + c_2\vec{w_2}c_3\vec{w_3} = c_1(r_{11}\vec{u_1} + r_{21}\vec{u_2} + r_{31}\vec{u_3}) + c_2(r_{12}\vec{u_1} + r_{22}\vec{u_2} + r_{32}\vec{u_3}) + c_3(r_{13}\vec{u_1} + r_{23}\vec{u_2} + r_{33}\vec{u_3}) = (c_1r_{11} + c_2r_{12} + c_3r_{13})\vec{u_1} + (c_1r_{21} + c_2r_{22} + c_3r_{23})\vec{u_2} + (c_1r_{31} + c_2r_{32} + c_3r_{33})\vec{u_3}$. So $[\vec{x}]_{\mathcal{U}} = \begin{bmatrix} c_1r_{11} + c_2r_{12} + c_3r_{13} \\ c_1r_{21} + c_2r_{22} + c_3r_{23} \\ c_1r_{31} + c_2r_{32} + c_3r_{33} \end{bmatrix} = R\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = R[x]_{\mathcal{W}}$. Notice that all this computation is the explicit computation of: $\vec{x} = \underbrace{\begin{bmatrix} | & | & | \\ \vec{w_1} & \vec{w_2} & \vec{w_3} \\ | & | & | \end{bmatrix}}_{P} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = P[x]_{\mathcal{W}} = SR[x]_{\mathcal{W}}$, Since $[\vec{x}]_{\mathcal{U}}$ is the vector, such that $\vec{x} = S[\vec{x}]_{\mathcal{U}}$, comparing both sides gives us $[\vec{x}]_{\mathcal{U}} = R[x]_{\mathcal{W}}$.

The matrix R is called the change of basis matrix, from basis $\mathcal{W}$ to basis $\mathcal{U}$. In order to remember the direction look at the formula:

$$[\vec{x}]_{\mathcal{U}} = R[\vec{x}]_{\mathcal{W}},$$

i.e. if you have the coordinates of $\vec{x}$ in $\mathcal{W}$ then you can get the coordinates in $\mathcal{U}$ by left multiplication by R. (this is also what the bottom paragraph on page 206 of your textbook refers to)