Math 19a - Reading 8.1
outline for discussion section

due Monday, February 25, 2008

1. How do you, guys, feel about vectors and matrices? Have you seen such things before? What did you think of the lecture on Monday? Today we will do another example of that and if you have any questions now is a good time to resolve them. Let's move on to the paper now. what did you think of it? Do you understand what's going on, what are the authors trying to show? We'll se now.

2. First of all, some background. The ozone layer is located in the lower portions of the stratosphere, right above the troposphere. It starts at 15-35 km above the Earth's surface, and it thickness varies geographically and seasonally. Ozone layer is important because it absorbs the Ultraviolet Radiation from the sun. In particular, the UV light is divided in three categories based on wavelength - UV-A, UV-B and UV-C. The UV-A rays contribute to the aging of the skin, while the UV-B rays cause genetic malformations, i.e. cancer, and UV-C is even more damaging. The ozone layer absorbs most of the UV-B light and all of the UV-C light.

   How does that all work? Oxygen is found in three forms: atomic, oxygen molecule and ozone. If an oxygen molecule is hit with UV light it is photolyzed into two atoms. Then an atom combines with a oxygen molecule to form ozone

   $$O_2 + h\nu \text{ (high } \nu) \to 2O \quad O_2 + O \to O_3 \quad O_3 + h\nu \text{ (low } \nu) \to O_2 + O.$$

   How are bromide and chlorine radicals harmful for the ozone.

   $$CFCl_3 + h\nu \to CFCl_2 + Cl$$

Then chlorine destroys ozone by the following reaction

$$Cl + O_3 \rightarrow ClO + O_2$$

$$CLO + O \rightarrow CL + O_2$$

I.e. the overall effect is to increase the rate of recombination from and speed the destruction of ozone. This mechanism works in the upper atmosphere where the free oxygen atoms are abundant, but there are more complicated mechanisms for the lower part that also lead to ozone destruction.

3. Now back to the paper. So far the harmfulness of chemicals has been measured by the statistics ODP (ozone depletion potential), but the author argue that this criterion should be refined by giving a few examples. How is ODP measured: the time-integrated (total so far) ozone depletion caused by a specific quantity of the chemical relative (divided by) same quantity of chlorofluorocarbon (CFC-11, chemically $CFCl_3$). The halocarbons (carbon atoms + halogen atoms) cycle between the troposphere and stratosphere, where a portion of the compound is dissociated producing CL or Br atoms. In the troposphere the HCFCs react with hydroxyl radicals (OH) and so are destroyed.

   The model is the following: Let B_T and B_S denote the amount of CL tied up in undissociated halocarbons in the troposphere/stratosphere respectively. Let C denote the quantity of free chlorine in stratosphere. Then the authors claim the following differential equations, system (1):

$$\begin{aligned}
\frac{dB_\tau}{dt} &= -\frac{B_\tau}{L_\tau} - \frac{B_\tau f - B_S}{\tau_t} = \left(-\frac{1}{L_\tau} - \frac{f}{\tau_t}\right) B_\tau + \frac{1}{\tau_t} B_S \\
\frac{dB_S}{dt} &= -\frac{B_S}{L_S} - \frac{B_S - B_\tau f}{\tau_t} = \frac{f}{\tau_t} B_\tau + \left(-\frac{1}{L_S} - \frac{1}{\tau_t}\right) B_S \\
\frac{dC}{dt} &= -\frac{C}{\tau_t} + \frac{B_S}{L_S}
\end{aligned}$$

   Note that the system can be written in matrix form as $\frac{d\vec{v}}{dt} = A\vec{v}$, where

$$\vec{v} = \begin{pmatrix} B_\tau \\ B_S \\ C \end{pmatrix} \text{ and } A = \begin{pmatrix} -\frac{1}{L_\tau} - \frac{f}{\tau_t} & \frac{1}{\tau_t} & 0 \\ \frac{f}{\tau_t} & -\frac{1}{L_S} - \frac{1}{\tau_t} & 0 \\ 0 & \frac{1}{B_S} & -\frac{1}{\tau_t} \end{pmatrix}.$$

4. Explain how the system can be understood as a two- (or three-) compartment model. For each "compartment" we have

$$(\text{rate of change}) = (\text{rate in}) - (\text{rate out}).$$

Consider the first equation for example. The rate in is represented by the stratospheric bound chlorine entering the troposphere. So it will be proportional to B_S with a constant $\frac{1}{\tau_t}$, whose meaning we will explain later.

For example, in the first equation, the "rate in" is $\frac{B_S}{\tau_t}$, which corresponds to chlorine moving from the stratosphere to the troposphere. More specifically, $\frac{B_S}{\tau_t}$ is equal to the rate r that air (in mass) moves from the stratosphere to the troposphere (and vice versa) times the concentration of chlorine in the stratosphere, which is B_S/M_S, where M_S is the total mass of air in the stratosphere. (So $\frac{r_S}{V_S} = \frac{1}{\tau_t}$.) They claim that the stratosphere is only 15% of the total air mass M, so $M_S = 0.15M$. Then since air is moving with the same rate upwards, the chlorine is leaving the troposphere at a rate of rB_T/M_T, but $M_T = 0.85M$, so we get $\frac{0.15}{0.85}\frac{1}{\tau_t}B_T$. This also explains the factor f.

Exactly analogously is the other equation derived. The third equation is derived similarly - the chlorine which moves from the stratosphere to the troposphere with the same air exchange and the creation of chlorine by the exact same degradation of the undissociated chlorine in the stratosphere.

5. Point out that the first two equations in (1) form their own two-dimensional system:

$$\begin{aligned}\frac{dB_\tau}{dt} &= \left(-\frac{1}{L_\tau} - \frac{f}{\tau_t}\right) B_\tau + \frac{1}{\tau_t} B_S \\ \frac{dB_S}{dt} &= = \frac{f}{\tau_t} B_\tau + \left(-\frac{1}{L_S} - \frac{1}{\tau_t}\right) B_S.\end{aligned}$$

Taking the given values $f = 0.15/0.85, \tau_t = 3, L_\tau = 1000$, and $L_s = 5$, and rounding to three significant digits we have

$$\begin{aligned}\frac{dB_\tau}{dt} &= -0.0598 B_\tau + 0.333 B_S \\ \frac{dB_S}{dt} &= 0.0588 B_\tau - 0.533 B_S.\end{aligned}$$

If $A = \begin{pmatrix} -0.0598 & 0.333 \\ 0.0588 & -0.533 \end{pmatrix}$, then $\det(A) = 0.0123 > 0$ and $\mathrm{tr}(A) = -0.593 < 0$. So our stability criterion predicts that the equilibrium at $(0, 0)$ is stable. What does this mean about the situation? Everything will eventually degrade /fall on Earth, as expected - there is no source here.

6. We already saw in class what eigenvalues and eigenvectors are, so let's briefly show how to use them here. If $v \neq 0$ is an eigenvector, then there is λ, s.t. $Av = \lambda v$, this is by definition, so we see that $(A - \lambda I)v =$, so the ratio of $\frac{-0.0598-\lambda}{0.333} = \frac{0.0588}{-0.533-\lambda}$, i.e. $(\lambda + 0.0598)(\lambda + 0.533) - 0.3330.0588 = 0$. This is actually the characteristic equation that we have in the handouts from last lecture. We can solve this quadratic equation and get two possible values for λ - $-0.0216 = -\frac{1}{46.5}$ and $-0.572 = -\frac{1}{1.75}$, which are exactly the exponents of the solutions in (2a) and (2b). We know that if $v = e^{\lambda_1 t}v_1$, then $\frac{dv}{dt} = e^{\lambda_1 t}\lambda_1 v_1 = Av_1 e^{\lambda_1 t} = Av$, so by superposition we should have the solutions of the form $a_1 e^{\lambda_1 t}v_1 + a_2 e^{\lambda_2 t}v_2$. So once we've found the eigenvalues we know to look for a solution of the type $B_S = a_1 v_1 e^{\lambda_1 t} + a_2 v_2 e^{\lambda_2 t}$ and we find a_1, a_2 from the initial conditions that they give in the paper.

   Once we have B_T and B_S we can solve for C using integrating factor.

7. We see that in the paper they call the constant L_S and L_T lifetimes of the chlorine compounds. Let $y(t)$ represent a quantity like population, amount of chlorine, ect that is governed by a simple exponential decay law. In general, the solution to an equation of the form $\frac{dy}{dt} = -cy, c > 0$, is $y(t) = y_0 e^{-ct}$. Then the total population over the whole time is $\int_0^\infty y(t)dt = y_0 \frac{1}{c}$. Then the probability density function of that population is $p(t) = \frac{y(t)}{y_0/c} = ce^{-ct}$. So for example, the probability that someone lives between α and β years is $\int\limits_\alpha^\beta p(t)dt$.

Then the average lifespan for a member of the population is then

$$\begin{aligned}
\int_{-\infty}^{\infty} tp(t)dt &= \int_{0}^{\infty} t(ce^{-ct})dt \\
&= \lim_{b\to\infty} \int_{0}^{b} cte^{-ct}dt \\
&= \lim_{b\to\infty} \left(-te^{-ct} - \frac{1}{c}e^{-ct}\right)_{0}^{b} \\
&= \lim_{b\to\infty} \left(-be^{-cb} - \frac{1}{c}e^{-cb} + 0 + \frac{1}{c}\right) \\
&= \frac{1}{c}.
\end{aligned}$$

This same concept came up in the HIV paper regarding the lifespans of cells and viruses.