Math 19a - Readings 2.2 and 6.1 outline for discussion section

due Wednesday, February 13, 2008

1 Left Snails and Right Minds

So what did you think about the article on snails? Yes, it actually doesn't tackle the issue about the abundance of right snails, but that's what we are concerned about here. So the article was just an entertaining reading to introduce us to the problem of too many right snails as opposed to left snails.

It's an interesting biological question why the left handed snails haven't died out, do they have any advantage in the world that keeps them alive? I found a recently published answer to that, it seems like their unusual orientation makes them difficult for predators to deal with them. In particular right-handed crabs can't really open a left-oriented shell. " Recent research, published in the Royal Society Biology Lettersby Yale professor Gregory Dietl and Jonathan Hendricks of the University of Kansas, reveals that left-handed snailsones with apertures on the left sides of their shellsare able to confound their predators, the crab Calappa flammea. The Yale researchers findings show that, much like right-handed athletes unaccustomed to dealing with their left-handed opponents, crabs find it difficult to eviscerate left-handed snails, opting instead to leave them be. "

So mu about the biology, let's now try to model the situation. We are going to focus on the two commentaries, given in chapters 2 and 5.

Let's consider chapter 2's commentary. First of all, what about the assumptions. The first one is not so surprising - all it says is that snails don't really distinguish among shell orientation - the probability of a snail mating with one of type A is proportional to the abundance of type A, as it is certainly true if type A is its own type. The second assumption about the offspring is also pretty natural (to be more precise one could consider right and left types as dominant and recessive alleles). (Don't say unless asked: There is a mistake however, namely the first paragraph on page 23 is not correct - the first assumption does not imply preference choice. In any case, that won't matter for our model.)

Now let's try to see where the model on page 23 comes from: Let R be number of right snails and L the number of left snails. Then $p(t)$ is the proportion of left-handed shells at time t, i.e. $p = \frac{L}{R+L}$. Now let's see what $\frac{dp}{dt}$ is:

$$\frac{dp}{dt} = \frac{d\frac{L}{L+R}}{dt} = \frac{\frac{dL}{dt}(L+R) - L\frac{d(L+R)}{dt}}{(L+R)^2} = \frac{\frac{dL}{dt}R - \frac{dR}{dt}L}{(L+R)^2}.$$

Now, according to the assumptions the new right offspring will be proportional to $L^2 + 1/2LR$ (number of pairs of type L, L and half of type L, R, similar to the third paragraph on page 23), and similarly the left offspring, so we can assume a proportionality constant β and then rewrite the above equation as

$$\frac{dp}{dt} = \beta\frac{(L^2 + 1/2LR)R - L(R^2 + 1/2LR)}{(L+R)^2} =$$

$$\beta\frac{1/2L^2R - 1/2LR^2}{(L+R)^2} = \frac{\beta}{2}\frac{L}{L+R}\frac{R}{L+R}(L-R) = \frac{\beta}{2}p(1-p)(L+R)(2p-1).$$

So taking $\alpha = \beta(R+L)/2$ gives the desired equation

$$\frac{dp}{dt} = \alpha p(1-p)(p-1/2).$$

Here we assume the total number of snails remains constant.

Do you remember this type of equation. The fisheries papers, depensation, modified logistic model. It's really the same. We analyze it the same way to find stable and unstable equilibria, so we see that if $p < 1/2$, so left snails are less than right snails, then p goes to 0, so left snails disappear, and similarly if $p > 1/2$. This is a positive feedback and it shows the behavior we want.

Now let's go to the model proposed in the commentary of chapter 5. It is a competing species model - assuming left and right snails fight which other for the same resources but don't kill each other. The model they propose is the following: Let $L(t)$ and $R(t)$ denote the populations of left- and right-handed snails, respectively, at time t. Then

$$\begin{aligned}\frac{dR}{dt} &= (R - R^2) - aRL \\ \frac{dL}{dt} &= (L - L^2) - aLR\end{aligned} \tag{1}$$

Here if there was no interaction the right snails's growth would be governed by a logistic equation. The interaction with the right snails has a negative effect and is proportional to both R and L. Let's analyze it, though it is analyzed in the book already. How do you guys feel about phase plane analysis. Can we do that model real quick for $a < 1$ and then $a > 1$?

Consider the case $a < 1$. What are the R null clines, i.e. when is $\frac{dR}{dt} = 0$: $R = 0$ so the L-axis and the line $1 - R - aL = 0$ which passes through $(0, 1/a)$ and $(1, 0)$. Similarly the L-null clines are $L = 0$ and $1 - L - aR = 0$. At the R null clines the corresponding derivative is 0, so the R is not changing, movement is only vertical. Let's see what directions it is: Figure 5.5 on page 79. We see that wherever we start we approach the equilibrium point with $R = L \neq 0$, which is $(\frac{1}{1+a}, \frac{1}{1+a})$. Does this describe the real world situation? No.

Now consider the other case $a > 1$. The null clines are the same, but the intersection points with the axes are exchanged, so the relative position of the null clines is reversed. Figure 5.2, 5.3, 5.4 describe our situation. We see now that there are two equilibria, at $(1, 0)$ and $(0, 1)$, so does this model describe our situation? Yes.

Now let's consider the next reading for chapter 6.

2 Red Grouse and Their Predators

1. What did you think of the paper? Where is the math here? Well, not explicitly but at least they provide us with this beautiful clear graph, the publishers were cheap on color ink. What we can get out of it is that grouse and raptor's population follow cycles as in our hare and lynx model, until the last ten years when grouse population has only been declining. Let's consider the Lotka-Volterra predator prey model:

2. Recall the Lotka-Volterra predator-prey model

$$\begin{aligned}\frac{dx}{dt} &= (a - bx - cy)x \\ \frac{dy}{dt} &= (-d + ex)y,\end{aligned}$$

where x represents the prey and y the predators. In class we drew the phase plane for this model and saw that the solutions circulate around an equilibrium. Although we cannot prove this yet, it turns out that the solutions spiral in, unless $b = 0$, in which case they are periodic (recall the hare and lynx model from the first day). In the situation described in the reading, the grouse population appears to have been reasonably periodic in the past, with big swings between "boom" years and "bust" years. This was when the raptor and harrier populations were low, or even nonexistent. Can the students conjecture a reason for this cyclic behavior? Would hunting account for it? Does it make sense that the "population" of hunters might also cycle, like that of the predator in the model above? Are there other predators? Remember - if the grouse population gets very low, the species (including humans) that prey on it would also decrease, hunters for example will start shooting somewhere else where they would more likely hit something. So this leaves grouse in peace for a while and their numbers start increasing. Then hunters would notice this and return to decrease this population.

3. Suppose the grouse population and the population of their primary predator behave according to the following model, with $b = 0$ since we already noticed that x has been periodic

$$\begin{aligned}\frac{dx}{dt} &= (a - cy)x \\ \frac{dy}{dt} &= (-d + ex)y.\end{aligned}$$

Now we introduce a new predator (raptors or harriers), who prey on many different species, and whose population is constant relative to the population of grouse. We could modify our model by introducing an additional term $-hx$ into the first equation, to account for the additional predation.

$$\begin{aligned}
\frac{dx}{dt} &= (a - cy)x - hx \\
\frac{dy}{dt} &= (-d + ex)y
\end{aligned}$$

Note that, unlike the situation of our logistic model with harvesting, the number of grouse caught by raptors is proportional to the population of grouse. (Why is a reasonable assumption?) If $h < a$, this model still has periodic solutions, although the equilibrium population of grouse will be smaller. However, if $h > a$, i.e., if the raptors are too effective, the population of grouse goes to zero.