Average lifespan in exponential decay

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We have seen in few of the papers so far implied statements of the kind: If c is the death(decay,degradation,dissociation) rate of a population (chemical concentration, etc), then $\frac{1}{c}$ is the average lifespan of that population. Having decay rate c, implies that if $y(t)$ is the population size at time t then $\frac{dy}{dt} = -cy$ intrinsically for the population (without any outside intervention, like predators,air convection etc). Based on this information we find the average lifespan of individuals from this population.

In general, the average value of a function $f(x)$ of x (where x is random) is the sum

$$\sum_{n=-\infty}^{\infty} f(n)Prob(x = n),$$

i.e. we sum $f(x)$ over all possible outcomes for the values of x. For example if $f(x) = x^2$ and $x = 1$ with probability 1/3 and $x = 2$ with probability 2/3 (so x can't be anything else), then the average (expected) value of $f(x)$ is

$$avg(f(x)) = f(0)\frac{1}{3} + f(1)\frac{2}{3} = 1 \cdot \frac{1}{3} + 4 \cdot \frac{2}{3} = 3.$$

This was in the discrete case, in the continuous case we need to pass from sums to integrals. We'll pass through Riemann sums: If $P(m)$ denotes the probability of $x < m$, then the probability that $x \in (x_{i-1}, x_i)$ is just $P(x_i) - P(x_{i-1})$ and so the average value for $f(x)$ would be the limit

$$\lim_{n\to\infty} \sum_{i=0}^{n} f(x_i)(P(x_i) - P(x_{i-1})) = \tag{1}$$

$$\lim_{n\to\infty} \sum_{i=0}^{n} f(x_i)\frac{P(x_i) - P(x_{i-1})}{x_i - x_{i-1}}(x_i - x_{i-1}) = \tag{2}$$

$$\int_{-\infty}^{+\infty} f(x)\frac{dP(x)}{dx}dx. \tag{3}$$

Thus the last expression ($\int_{-\infty}^{+\infty} f(x)\frac{dP(x)}{dx}dx$) is in fact the definition of average value of $f(x)$ in the continuous case.

Back to our problem, if $\frac{dy}{dt} = -cy$, what is the average lifespan of the population. We need to find the probability that an individual lives less than T years - this would be the portion of the population that died before $t = T$, or in other words it will be 1 minus the portion of the population that lived longer than $t = t$ years, that is actually $y(T)$ (number of individuals who survived until T). So $P(T) = 1 - \frac{y(T)}{y(0)}$ as $y(0)$ is the total initial population, when $T > 0$, if $T < 0$ then $P(T) = 0$ (nobody lives negative number of years). Note that the exponential decay equation we have has the simple solution $y(t) = y(0)e^{-ct}$, so $P(T) = 1 - e^{-cT}$.The function we want to find the average of is just $f(t) = t$, so we plug $x = t$, $f(x) = t$ and the probability we just found into the above formula:

$$\text{average lifespan } = \int_{t=-\infty}^{+\infty} t\frac{dP(t)}{dt}dt = \tag{4}$$

$$\int_{t=0}^{+\infty} t\frac{d(1-e^{-ct})}{dt}dt = \tag{5}$$

$$\int_{t=0}^{+\infty} t(+ce^{-ct})dt = \tag{6}$$

$$\int_{t=0}^{+\infty} -td(e^{-ct}) = -te^{-ct}\Big|_0^\infty - \int_0^\infty -e^{-ct}dt = \tag{7}$$

$$(0-0) - (\frac{1}{c}e^{-ct})\Big|_0^\infty = -(0-\frac{1}{c}) = \frac{1}{c}. \tag{8}$$

So the average lifespan is in fact $\frac{1}{c}$ years, which is the same as saying $c = \frac{1}{\text{avg lifespan}}$.