Littlewood Paley Construction

1) Let $C = \{ \frac{3}{4} \le |\xi| \le \frac{8}{3} \}$ and $B = \{ 0 \le |\xi| \le \frac{4}{3} \}$

2) Let $\alpha \in (1, \frac{4}{3})$. Then $C' = \{ \frac{1}{\alpha} \le |\xi| \le 2\alpha \} \subsetneq C$

3) Let $\theta \in C_c^\infty(\mathbb{R}^n)$ be defined so that $\operatorname{supp}\theta \subsetneq C$, $\theta \equiv 1$ on C'

and $0 \le \theta \le 1$.

4) Take $S(\xi) = \sum_{j \in \mathbb{Z}} \theta(2^{-j}\xi)$. Since $\operatorname{supp}\theta(2^{-j}\xi) \cap \operatorname{supp}\theta(2^{-j'}\xi) \neq \phi$

iff $|j-j'| \le 1$, $S(\xi)$ converges for each ξ and $S(\xi) > 0$. $(\xi \neq 0)$

5) Let $\varphi(\xi) = \frac{\theta(\xi)}{S(\xi)}$. Then $\varphi(2^{-j}\xi) = \frac{\theta(2^{-j}\xi)}{\sum_{j' \in \mathbb{Z}} \theta(2^{-j'}2^{-j}\xi)} = \frac{\theta(2^{-j}\xi)}{\sum_{j' \in \mathbb{Z}} \theta(2^{-j'}\xi)}$

$\Rightarrow \varphi(2^{-j}\xi) = \frac{\theta(2^{-j}\xi)}{S(\xi)}$. Thus, $\sum_{j \ge 0} \varphi(2^{-j}\xi) = \frac{\sum_{j \ge 0} \theta(2^{-j}\xi)}{S(\xi)}$

6) If $|\xi| \ge 4/3$ then $\theta(2^{-j}\xi) = 0$ for $j < 0$. Thus, $S(\xi) = \sum_{j \ge 0} \theta(2^{-j}\xi)$

Hence, for $|\xi| \ge 4/3$, $\sum_{j \ge 0} \varphi(2^{-j}\xi) = 1$.

7) Define $\chi(\xi) = 1 - \sum_{j \ge 0} \varphi(2^{-j}\xi)$

8) Thus, $\operatorname{supp}\varphi(2^{-j}\cdot) = \operatorname{supp}\theta(2^{-j}\cdot) \subsetneq \{ 2^j \frac{3}{4} \le |\xi| \le 2^{j+1}\frac{4}{3} \}$

and $\operatorname{supp}\chi(2^{-j}\cdot) \subsetneq \{ 0 \le |\xi| \le 2^j \frac{4}{3} \}$.

9) Hence ① $\operatorname{supp}\varphi(2^{-j}\cdot) \cap \operatorname{supp}\varphi(2^{-j'}\cdot) = \phi$ iff $|j-j'| > 1$.

and ② $\operatorname{supp}\varphi(2^{-j}\cdot) \cap \operatorname{supp}\chi = \phi$ iff $j \ge 1$.

③ $\forall \xi \in \mathbb{R}^d$ $\chi(\xi) + \sum_{j \ge 0} \varphi(2^{-j}\xi) = 1$

④ $\forall \xi \in \mathbb{R}^d \setminus \{0\}$ $\sum \varphi(2^{-j}\xi) = 1$.

Turn Over

10) Further, it is easy to check that if $\tilde{C} = B(0, 2/3) + C$ then

$$2^j \tilde{C} \cap 2^{j'} C = \phi \quad \text{when } |j-j'| \geq 5$$

11) Since $0 \leq \chi \leq 1$, $0 \leq \varphi \leq 1$, it is clear that $\chi^2(\xi) + \sum_{j \geq 0} \varphi^2(2^{-j}\xi) \leq 1$. from ③ in 2). On the other hand, $1 = 1^2 = \left(\sum_{\substack{j \in 2\mathbb{Z} \\ j \geq 0}} \varphi(2^{-j}\xi) + \chi(\xi) + \sum_{\substack{j \notin 2\mathbb{Z} \\ j \geq 0}} \varphi(2^{-j}\xi) \right)^2$

Thus, $1 \leq 2\left(\sum_{\substack{j \in 2\mathbb{Z} \\ j \geq 0}} \varphi(2^{-j}\xi) \right)^2 + 2\left(\chi(\xi) + \sum_{\substack{j \notin 2\mathbb{Z} \\ j \geq 0}} \varphi(2^{-j}\xi) \right)^2$ by Young's Ineq.

But since $\operatorname{supp} \varphi(2^{-j}\xi) \cap \operatorname{supp} \varphi(2^{-j \pm 2}\xi) = 0$ and similar for χ and $\varphi(2^{-j}\xi)$,

we get $\frac{1}{2} \leq \sum_{\substack{j \in 2\mathbb{Z} \\ j \geq 0}} \varphi(2^{-j}\xi)^2 + \chi(\xi)^2 + \sum_{\substack{j \notin 2\mathbb{Z} \\ j \geq 0}} \varphi(2^{-j}\xi)^2$

$$= \chi(\xi)^2 + \sum_{j \geq 0} \varphi(2^{-j}\xi)^2$$

12) Thus, ⓐ $\frac{1}{2} \leq \chi(\xi)^2 + \sum_{j \geq 0} \varphi(2^{-j}\xi)^2 \leq 1$

ⓑ $\frac{1}{2} \leq \sum_{j \in \mathbb{Z}} \varphi(2^{-j}\xi)^2 \leq 1$

Inhomogeneous Littlewood Paley: $\Delta_j u = 0$ if $j \leq -2$

$$\Delta_{-1} u = \chi(D) u$$

$$\Delta_j u = \varphi(2^{-j}D) u \quad j \geq 0$$

$$S_j u = \sum_{j' \leq j-1} \Delta_{j'} u$$

Homogeneous L-P: $\dot{\Delta}_j u = \varphi(2^{-j}D) u \quad j \in \mathbb{Z}$

$$\dot{S}_j u = \chi(2^{-j}D) u \quad j \in \mathbb{Z}.$$

For $u \in S_h'$, we can write $u = \sum_{j \in \mathbb{Z}} \dot{\Delta}_j u$ and $\dot{S}_j u = \sum_{j' \leq j-1} \dot{\Delta}_{j'} u$

and that $\lim_{j \to -\infty} \dot{S}_j u = 0$.

Besov Spaces

Def: Say $s \in \mathbb{R}$ and $r, p \in [1, \infty]$. Then define the homogeneous Besov space $\dot{B}^s_{p,r}$ as the space of $u \in \mathcal{S}_h'$ such that

$$\|u\|_{\dot{B}^s_{p,r}} = \left\| 2^{js} \|\dot{\Delta}_j u\|_{L^p} \right\|_{\ell^r(\mathbb{Z})} < \infty. \quad \text{where } j \in \mathbb{Z}$$

This is a normed space. A quick calculation shows that it is a seminorm.

If $\|u\|_{\dot{B}^s_{p,r}} = 0$ then $\|\dot{\Delta}_j u\|_{L^p} = 0 \ \forall j \in \mathbb{Z}$. Thus,

$\dot{\Delta}_j u \equiv 0 \ \forall j \in \mathbb{Z}$. Now, since $u \in \mathcal{S}_h'$, we have that

$\sum_{j \in \mathbb{Z}} \dot{\Delta}_j u = u$ ($u \in \mathcal{S}_h'$ means that $\lim_{j' \to -\infty} \dot{S}_{j'} u = 0$.

Thus, since $\dot{S}_{j'} u = \chi(2^{-j'} D) u = \left(1 - \sum_{j \geq 0} \varphi(2^{-j} 2^{-j'} D)\right) u$

$$= \left(1 - \sum_{j \geq -j'} \varphi(2^{-j} D)\right) u = u - \sum_{j \geq j'} \varphi(2^{-j} D) u = u - \sum_{j \geq j'} \dot{\Delta}_j u$$

we send $j' \to -\infty$ to get that $0 = u - \sum_{j \in \mathbb{Z}} \dot{\Delta}_j u$)

Thus, $u = 0$.

Remark: $\|\cdot\|_{\dot{B}^s_{p,r}}$ is independent of choice of χ, φ:

Say $\tilde{\chi}, \tilde{\varphi}$ are two other choices that satisfy L-P decomp. conditions.

Then $2^{js} \|\tilde{\varphi}(2^{-j} D) u\|_{L^p} = 2^{js} \left\| \tilde{\varphi}(2^{-j} D) \left(\sum_{j' \in \mathbb{Z}} \dot{\Delta}_{j'} u\right) \right\|_{L^p}$

$$\leq \sum_{j' \in \mathbb{Z}} 2^{js} \|\dot{\Delta}_{j'} \tilde{\varphi}(2^{-j} D) u\|_{L^p} \leq \sum_{|j'-j| \leq N_0} 2^{js} \|\dot{\Delta}_{j'} \tilde{\varphi}(2^{-j} D) u\|_{L^p} \quad \left(\begin{matrix}\text{for} \\ \text{some} \\ N_0\end{matrix}\right)$$

$$\leq C 2^{|N_0| s} \sum_{j' \in \mathbb{Z}} 1_{[-N_0, N_0]}(j-j') \, 2^{j's} \|\dot{\Delta}_{j'} u\|_{L^p} \quad \text{since } 2^{j's} 2^{N_0 |s|} \geq 2^{js}$$

Now, apply Young's Inequality to get

$$\| 2^{js} \| \tilde{\varphi}(2^{-j}D) u \|_{L^p} \|_{\ell^r} \leq C 2^{N_0 |s|} \| 1_{[-N_0, N_0]} \|_{\ell^1} \| u \|_{\dot{B}^s_{p,r}}$$

<u>Prop</u>: $\dot{H}^s \subseteq \dot{B}^s_{2,2}$ $\forall s \in \mathbb{R}$.

<u>Proof</u>: $\| u \|_{\dot{B}^s_{2,2}} = \| 2^{js} \| \dot{\Delta}_j u \|_{L^2} \|_{\ell^2} = \| 2^{js} \| \varphi(2^{-j}\xi) \hat{u}(\xi) \|_{L^2} \|_{\ell^2}$

$$\leq \| 2^{js} \| \varphi(2^{-j}\xi) |\xi|^{-s} \|_{L^\infty} \| u \|_{\dot{H}^s} \|_{\ell^2}.$$

$$= \| u \|_{\dot{H}^s} \| 2^{js} \| \varphi(2^{-j}\xi) |\xi|^{-s} \|_{L^\infty(2^{-j}C)} \|_{\ell^2}$$

$$\leq \| u \|_{\dot{H}^s} \| 2^{js} \cdot 2^{-js} \cdot \frac{8}{3} \| \varphi(2^{-j}\xi) \|_{L^\infty} \|_{\ell^2}$$

$$\leq C \| u \|_{\dot{H}^s} \| \| \varphi(2^{-j}\xi) \|_{L^\infty} \|_{\ell^2} \lesssim \| u \|_{\dot{H}^s}$$ (by (2.11) Bahouri et al page 59)

Further, if $u \in \dot{H}^s$ then $u \in \mathcal{S}'(\mathbb{R}^d) \cap \dot{H}^s(\mathbb{R}^d)$ so why should $u \in \mathcal{S}'_h(\mathbb{R}^d)$? Because $\hat{u}(\xi)$ decays faster than $|\xi|^s$ at origin.

Then $\lim_{\lambda \to \infty} \| \theta(\lambda D) u \|_{L^\infty} = \lim_{\lambda \to \infty} \| \mathcal{F}^{-1} [\theta(\lambda\xi) \hat{u}(\xi)] \|_{L^\infty} \leq \lim_{\lambda \to \infty} \| \theta(\lambda\xi) \hat{u}(\xi) \|_{L^1}$

$\theta(\xi) = |\xi|^{-s} \chi(\xi)$

compactly supported around origin.

$$\leq \lim_{\lambda \to \infty} \lambda^s \| |\xi|^s \hat{u}(\xi) \|_{L^2} \| \chi(\lambda\xi) \|_{L^2}$$

Fact : The norms $\|\cdot\|_{\dot{H}^s}$ and $\|\cdot\|_{\dot{B}^s_{2,2}}$ are equivalent.

Proof:

First, by Minkowski's Inequality, $\Big\| \|f(t,x)\|_{L^2_t} \Big\|_{L^2_x} \leq \Big\| \|f(t,x)\|_{L^2_x} \Big\|_{L^2_t}$

and $\Big\| \|f(t,x)\|_{L^2_x} \Big\|_{L^2_t} \leq \Big\| \|f(t,x)\|_{L^2_t} \Big\|_{L^2_x}$. So $\Big\| \| \; \|_{L^2_t} \Big\|_{L^2_x} = \Big\| \| \; \|_{L^2_x} \Big\|_{L^2_t}$

So $$\|u\|_{\dot{B}^s_{2,2}} = \Big\| 2^{js} \|\dot{\Delta}_j u\|_{L^2} \Big\|_{\ell^2} = \Big\| 2^{js} \|\varphi(2^{-j}\xi)\hat{u}(\xi)\|_{L^2} \Big\|_{\ell^2}$$

$$= \Big\| \| 2^{js} \varphi(2^{-j}\xi) |\xi|^{-s} |\xi|^s \hat{u}(\xi) \|_{L^2} \Big\|_{\ell^2}$$

(up to a constant of $[\frac{1}{2}, 2]$) $$\approx \Big\| \| 2^{js} \varphi(2^{-j}\xi) 2^{-js} |\xi|^s \hat{u}(\xi) \|_{L^2} \Big\|_{\ell^2}$$

$$= \Big\| \|\varphi(2^{-j}\xi)\|_{\ell^2} |\xi|^s \hat{u}(\xi) \Big\|_{L^2} \leq \Big\| \|\varphi(2^{-j}\xi)\|_{\ell^2} \Big\|_{L^\infty} \|u\|_{\dot{H}^s}$$

Now, $$\|\varphi(2^{-j}\xi)\|^2_{\ell^2} = \sum_{j\in\mathbb{Z}} \varphi(2^{-j}\xi)^2 \leq \Big[\sum_{j\in\mathbb{Z}} \varphi(2^{-j}\xi)\Big]^2 \leq 1^2 = 1$$

since $0 \leq \varphi \leq 1$.

also, $$\Big[\sum_{j\in\mathbb{Z}} \varphi(2^{-j}\xi)\Big]^2 = \Big[\sum_{j \text{ odd}} \varphi(2^{-j}\xi) + \sum_{j \text{ even}} \varphi(2^{-j}\xi)\Big]^2$$

$$\leq 2\Big[\Big(\sum_{j \text{ odd}} \varphi(2^{-j}\xi)\Big)^2 + \Big(\sum_{j \text{ even}} \varphi(2^{-j}\xi)\Big)^2\Big]$$

$$\leq 2\sum_{j \text{ odd}} \varphi(2^{-j}\xi)^2 + 2\sum_{j \text{ even}} \varphi(2^{-j}\xi)^2$$

$$\leq 2\sum_{j\in\mathbb{Z}} \varphi(2^{-j}\xi)^2$$

Now, if $u \in \dot{H}^s$ then $\hat{u} \in L^1_{loc}$ so

$$\lim_{\lambda\to\infty} \| \theta(\lambda D) u \|_{L^\infty} \le \lim_{\lambda\to\infty} \| \theta(\lambda\xi)\, \hat{u}(\xi) \|_{L^1} = \lim_{\lambda\to\infty} \| \theta(\lambda\xi)\, \hat{u}(\xi) \|_{L^1(\frac{\mathrm{supp}\,\theta}{\lambda})}$$

$$\le \lim_{\lambda\to\infty} C \| \hat{u}(\xi) \|_{L^1(\frac{\mathrm{supp}\,\theta}{\lambda})} \longrightarrow 0 \text{ as } \lambda \to \infty.$$

$\Rightarrow u \in S'_h \Rightarrow u \in \dot{B}^s_{2,2}$

Conversely, say $s < d/2$ and $u \in \dot{B}^s_{2,2}$. Then $u \in S'_h$. We have to show $\hat{u} \in L^1_{loc}$. Say $K \subseteq \mathbb{R}^d$ compact. Then

$$\int_K |\hat{u}(\xi)|\, d\xi \le \int_K |\xi|^{-s} |\xi|^s |\hat{u}(\xi)|\, d\xi \le \| |\xi|^{-s} \|_{L^2} \| u \|_{\dot{H}^s}$$

$$\lesssim \int_{K \subseteq \mathbb{R}^d} |\xi|^{-2s}\, d\xi \quad \| u \|_{\dot{B}^s_{2,2}}$$

$$\lesssim \int_0^R r^{-2s} r^{d-1}\, dr < \infty.$$

$\Rightarrow u \in \dot{H}^s$

Prop: $\exists\ C_s$ such that $C_s^{-1} \lambda^{s-\frac{d}{p}} \|u\|_{\dot{B}^s_{p,r}} \le \|u(\lambda\cdot)\|_{\dot{B}^s_{p,r}} \le C_s \lambda^{s-\frac{d}{p}} \|u\|_{\dot{B}^s_{p,r}}$

Proof: Let $\tilde{\varphi}(\cdot) = \varphi(2^N \lambda^{-1} \cdot)$ where $2^N \le \lambda \le 2^{N+1}$

$$\|\dot{\Delta}_j(u(\lambda\cdot))\|_{L^p} \le C_s \|\tilde{\dot{\Delta}}_j(u(\lambda\cdot))\|_{L^p}$$ by Remark on I s.t. $C_s \in \ell^1$???

where $\tilde{\dot{\Delta}}_j u = \mathcal{F}^{-1}[\tilde{\varphi}(2^{-j}\cdot)] * u(\lambda\cdot)$

$$= 2^{jd-Nd}\lambda^d \int_{\mathbb{R}^n} h(2^{j-N}\lambda y)\, u(\lambda x - \lambda y)\, dy$$

$$= 2^{jd-Nd}\lambda^d \int_{\mathbb{R}^n} h(2^{j-N}\tilde{y})\, u(\lambda x - \tilde{y})\, \frac{d\tilde{y}}{\lambda^d}$$

$$= 2^{jd-Nd} \int_{\mathbb{R}^n} h(2^{j-N}\tilde{y})\, u(\lambda x - \tilde{y})\, d\tilde{y}$$

$$= \dot{\Delta}_{j-N} u(\lambda x)$$

$$\Rightarrow \|\dot{\Delta}_j u(\lambda\cdot)\|_{L^p} \le C_s \lambda^{-d/p} \|\dot{\Delta}_{j-N} u\|_{L^p}$$

$$\Rightarrow 2^{js} \|\dot{\Delta}_j u(\lambda\cdot)\|_{L^p} \le C_s \lambda^{-\frac{d}{p}} 2^{(j-N)s} 2^{Ns} \|\dot{\Delta}_{j-N} u\|_{L^p}$$

$$\le C_s \lambda^{s-\frac{d}{p}} 2^{(j-N)s} \|\dot{\Delta}_{j-N} u\|_{L^p}.$$

Taking ℓ^r norms, $\|u(\lambda\cdot)\|_{\dot{B}^s_{p,r}} \le C_s \lambda^{s-\frac{d}{p}} \|u\|_{\dot{B}^s_{p,r}}$

Other direction is similar.

$\blacksquare$

Prop: $1 \le p_1 \le p_2 \le \infty$ and $1 \le r_1 \le r_2 \le \infty$. Then $\forall\ s \in \mathbb{R}$,

$$\dot{B}^s_{p_1,r_1} \hookrightarrow \dot{B}^{s-d(\frac{1}{p_1}-\frac{1}{p_2})}_{p_2,r_2} \quad \text{continuously.}$$

Proof: $\|\dot{\Delta}_j u\|_{L^{p_2}} \le C 2^{jd(\frac{1}{p_1}-\frac{1}{p_2})} \|\dot{\Delta}_j u\|_{L^{p_1}}$ by Bernstein's Ineq.

Since $\ell^{r_1} \hookrightarrow \ell^{r_2}$ continuous, we are done. ∎

(If $\|a_n\|_{\ell^{r_1}} = 1$ then $|a_n| \le 1\ \forall n \Rightarrow \underset{\text{since } r_1 < r_2}{\sum |a_n|^{r_2} \le \sum |a_n|^{r_1}} \le 1$.)

Ex: Find a homogeneous function that is in some $\dot{B}^s_{p,r}$ but not in $\dot{H}^s$ or L^p?

[Let $\rho_\sigma = |\cdot|^{-\sigma}$. Thus, $\rho_\sigma \notin L^p(\mathbb{R}^d)\ \forall p \in [1,\infty]$,

and hence, not in $W^{k,p}(\mathbb{R}^d)\ \forall\ k,p$. Say $\sigma \in (0,d)$.

Let $\rho_\sigma = \rho_0 + \rho_1$ where $\rho_0 = \chi \rho_\sigma$, $\rho_1 = (1-\chi)\rho_\sigma$.

with $\chi \in C_c^\infty(\mathbb{R}^d)$ and $\chi \equiv 1$ near $B(0,1)$ unit ball.

Then, $\rho_0 \in L^1$, $\rho_1 \in L^q\ \forall\ q \in (\frac{d}{\sigma}, \infty)$. Thus, $\rho_\sigma \in S_h'$.

Furthermore, $\dot{\Delta}_j \rho_\sigma = 2^{jd} \rho_\sigma * h(2^j \cdot) = 2^{j(d+\sigma)} \rho_\sigma(2^j \cdot) * h(2^j \cdot)$

$$= 2^{j\sigma} (\rho_\sigma * h)(2^j x) = 2^{j\sigma} \dot{\Delta}_0 \rho_\sigma (2^j \cdot)$$

$\Longrightarrow \|\dot{\Delta}_j \rho_\sigma\|_{L^1} = 2^{j(\sigma-d)} \|\dot{\Delta}_0 \rho_\sigma\|_{L^1}$. Thus,

$$\left\| 2^{j(d-\sigma)} \|\dot{\Delta}_j \rho_\sigma\|_{L^1} \right\|_{\ell^\infty} = \|\rho_\sigma\|_{\dot{B}^{d-\sigma}_{1,\infty}} \le \|\dot{\Delta}_0 \rho_\sigma\|_{L^1}$$

Since $\|\dot{\Delta}_0 \rho_\sigma\|_{L^1} \le \|\dot{\Delta}_0 \rho_0\|_{L^1} + \|\dot{\Delta}_0 \rho_1\|_{L^1} \lesssim \|\rho_0\|_{L^1} + \|\dot{\Delta}_0 \rho_1\|_{L^q}$

$\lesssim \|\rho_0\|_{L^1} + \|\rho_1\|_{L^q}$ for $q \in (\frac{d}{\sigma}, \infty)$ by Bernstein's Ineq and since $\dot{\Delta} : L^p \to L^p$ $(p \in [1,\infty]$

<u>Prop</u>: (Interpolation) Say $\theta \in (0,1)$ and $s_1 < s_2$ and $p, r \in [1,\infty]$.

1) $\|u\|_{\dot{B}^{s_1\theta + s_2(1-\theta)}_{p,r}} \leq \|u\|^{\theta}_{\dot{B}^{s_1}_{p,r}} \|u\|^{1-\theta}_{\dot{B}^{s_2}_{p,r}} \quad \forall\ u \in S_h'$

2) $\|u\|_{\dot{B}^{s_1\theta + s_2(1-\theta)}_{p,1}} \leq \dfrac{C}{s_2 - s_1}\left(\dfrac{1}{\theta} + \dfrac{1}{1-\theta}\right) \|u\|^{\theta}_{\dot{B}^{s_1}_{p,\infty}} \|u\|^{1-\theta}_{\dot{B}^{s_2}_{p,\infty}}$

<u>Proof</u>: 1) Simple application of Hölder's Inequality.

2)
$$\|u\|_{B^{s_1\theta+s_2(1-\theta)}_{p,1}} = \sum_{j \leq N} 2^{j(s_1\theta + s_2(1-\theta))}\|\dot{\Delta}_j u\|_{L^p} + \sum_{j > N} 2^{j(s_1\theta + s_2(1-\theta))}\|\dot{\Delta}_j u\|_{L^p}$$

$$\leq \|u\|_{\dot{B}^{s_1}_{p,\infty}} \sum_{j\leq N} 2^{j(1-\theta)(s_2-s_1)} + \|u\|_{\dot{B}^{s_2}_{p,\infty}} \sum_{j>N} 2^{-j\theta(s_2-s_1)}$$

$$\leq \|u\|_{\dot{B}^{s_1}_{p,\infty}} \frac{2^{N(1-\theta)(s_2-s_1)}}{2^{(1-\theta)(s_2-s_1)}-1} + \|u\|_{\dot{B}^{s_2}_{p,\infty}} \frac{2^{-N\theta(s_2-s_1)}}{1-2^{-\theta(s_2-s_1)}}$$

Now pick N s.t.
$$\frac{\|u\|_{\dot{B}^{s_1}_{p,\infty}}}{\|u\|_{\dot{B}^{s_2}_{p,\infty}}} \leq 2^{N(s_2-s_1)} < 2^{s_2-s_1}\frac{\|u\|_{\dot{B}^{s_1}_{p,\infty}}}{\|u\|_{\dot{B}^{s_2}_{p,\infty}}}$$

Thus
$$\|u\|_{B^{s_1\theta+s_2(1-\theta)}_{p,1}} \leq \|u\|^{\theta}_{\dot{B}^{s_1}_{p,\infty}} \frac{\|u\|^{1-\theta}_{\dot{B}^{s_2}_{p,\infty}}}{2^{(1-\theta)(s_2-s_1)}-1} + \|u\|^{1-\theta}_{\dot{B}^{s_2}_{p,\infty}} \frac{\|u\|^{\theta}_{\dot{B}^{s_1}_{p,\infty}}}{1-2^{-\theta(s_2-s_1)}}$$

$$= \left(\frac{1}{2^{(1-\theta)(s_2-s_1)}-1} + \frac{1}{1-2^{-\theta(s_2-s_1)}}\right)\|u\|^{\theta}_{\dot{B}^{s_1}_{p,\infty}}\|u\|^{1-\theta}_{\dot{B}^{s_2}_{p,\infty}}$$

$\blacksquare$

Lemma: Say $\mathcal{C}'$ annulus and $(u_j)_{j\in\mathbb{Z}}$ sequence s.t. $\operatorname{supp}\hat{u}_j \subset 2^j\mathcal{C}'$ and $\| 2^{js}\|u_j\|_{L^p}\|_{\ell^r} < \infty$. If $\sum_{j\in\mathbb{Z}} u_j$ converges in S' to some $u \in S'_h$ then $u \in \dot{B}^s_{p,r}$ and $\|u\|_{\dot{B}^s_{p,r}} \leq C_s \| 2^{js}\|u_j\|_{L^p}\|_{\ell^r}$.

Proof: $\exists N_0$ st. $|j-j'| \geq N_0 \Rightarrow \dot{\Delta}_{j'} u_j = 0$. Thus,

$$\|\dot{\Delta}_{j'} u\|_{L^p} = \|\sum_{|j-j'|<N_0} \dot{\Delta}_{j'} u_j\| \leq C \sum_{|j-j'|<N_0} \|u_j\|_{L^p}$$

$$\Rightarrow \left\|2^{j's}\|\dot{\Delta}_{j'}u\|_{L^p}\right\|_{\ell^r} \leq \left\|C\sum_{|j-j'|\leq N_0} 2^{j's}\|u_j\|_{L^p}\right\|_{\ell^r} = \left\|C\sum_{j\in\mathbb{Z}} 1_{[-N_0,N_0]}(j'-j)\cdot 2^{j's}\|u_j\|_{L^p}\right\|_{\ell^r}$$

$$\leq C\|1_{[-N_0,N_0]}\|_{\ell^1}\|2^{js}\|u_j\|_{L^p}\|_{\ell^r} \lesssim \|2^{js}\|u_j\|_{L^p}\|_{\ell^r}.$$

∎

Thm: (Fatou Property) Say $(s_1,s_2)\in\mathbb{R}^2$ and $1\leq p_1,p_2,r_1,r_2\leq\infty$ s.t. (s_1,p_1,r_1) satisfies $\left[s_1 < \frac{d_1}{p_1} \text{ or } s = \frac{d_1}{p_1} \text{ and } r_1 = 1 \right] *$

If $(u_n)_{n\in\mathbb{N}}$ bounded in $\dot{B}^{s_1}_{p_1,r_1}\cap\dot{B}^{s_2}_{p_2,r_2}$ then $\exists\, u\in\dot{B}^{s_1}_{p_1,r_1}\cap\dot{B}^{s_2}_{p_2,r_2}$ and subsequence $(u_{\psi(n)})_{n\in\mathbb{N}} \subseteq (u_n)_{n\in\mathbb{N}}$ s.t. $u_{\psi(n)} \to u$ in S' and $\|u\|_{\dot{B}^{s_k}_{p_k,r_k}} \leq C \liminf_{n\to\infty} \|u_{\psi(n)}\|_{\dot{B}^{s_k}_{p_k,r_k}}$ $(k=1,2)$.

Remark: $*$ condition on (s,p,r) is special. since if it holds, then by Bernstein's Ineq., for u_j as in the Lemma above, $\|u_j\|_{L^\infty} \leq 2^{j d/p}\|u_j\|_{L^p}$

So $\|\sum_{j<0} u_j\|_{L^\infty} \leq \sum_{j<0}\|u_j\|_{L^\infty} \leq \sum_{j<0} 2^{j d/p}\|u_j\|_{L^p} = \sum_{j<0} 2^{j(\frac{d}{p}-s)} 2^{js}\|u_j\|_{L^p}$

$$\leq \|2^{j(\frac{d}{p}-s)}\|_{\ell^{r'}(\mathbb{Z})}\|2^{js}\|u_j\|_{L^p}\|_{\ell^r} < \infty. \text{ Thus}$$

IV

$\sum_{j\in\mathbb{Z}} u_j$ converges to some $u \in S'$ since $\lim_{j\to-\infty} \sum_{j'<j} u_{j'} = 0$ in L^∞.

and $\dot{S}_j u = \sum_{j'<j+N_0} u_{j'} \longrightarrow 0$ as $j \to -\infty \Longrightarrow u \in S_h'$. So the assumptions of convergence are taken care of.

<u>Thm</u>: (Fatou's Property) Say $(s_1,s_2)\in\mathbb{R}^2$ and $1\le p_1,p_2,r_1,r_2\le\infty$ and $s_1<d/p_1$.

If $(u_n)_{n\in\mathbb{N}}$ bounded sequence in $\dot{B}^{s_1}_{p_1,r_1}\cap\dot{B}^{s_2}_{p_2,r_2}$, then $\exists$ $u\in\dot{B}^{s_1}_{p_1,r_1}\cap\dot{B}^{s_2}_{p_2,r_2}$ and subsequence $(u_{\psi(n)})_{n\in\mathbb{N}}$ s.t. $\lim_{n\to\infty} u_{\psi(n)} = u$ in S' and

$$\|u\|_{\dot{B}^{s_k}_{p_k,r_k}} \le C \liminf_{n\to\infty} \|u_{\psi(n)}\|_{\dot{B}^{s_k}_{p_k,r_k}} \quad \text{for } k=1,2.$$

<u>Proof</u>: By Bernstein's Inequality for fixed $j\in\mathbb{Z}$, $(\dot{\Delta}_j u_n)_{j\in\mathbb{N}}$ is bounded in $L^{\min(p_1,p_2)}\cap L^\infty$. $\left[\ \|\dot{\Delta}_j u_n\|_{L^\infty} \le 2^{jd/p}\|\dot{\Delta}_j u\|_{L^p}\ \right]$.

Thus, $\liminf_{n\to\infty}\|\dot{\Delta}_j u_n\|_{L^{p_k}}$ exists and $\exists$ subsequence $\dot{\Delta}_j u_{\tilde{\psi}_j(n)}$ s.t. $\lim_{n\to\infty}\|\dot{\Delta}_j u_{\tilde{\psi}_j(n)}\|_{L^{p_k}} = \liminf_{n\to\infty}\|\dot{\Delta}_j u_n\|_{L^{p_k}}$. Furthermore, this gives a subsequence of $\tilde{\psi}_j(n)$ that weakly convergences: $\lim_{n\to\infty}\langle\dot{\Delta}_j u_{\tilde{\psi}_j(n)},\phi\rangle = \langle\tilde{u}_j,\phi\rangle$ for some $\tilde{u}_j\in L^{p_k}$. $\forall$ $\phi\in S$ (since $S(\mathbb{R}^d)\in L^{p_k'}(\mathbb{R}^d)$)

Note that $\tilde{u}_j\in S'(\mathbb{R}^d)$ since $\sup_{\|\phi\|_{L^{p_k'}}=1}\langle\tilde{u}_j,\phi\rangle \le \lim_{n\to\infty}\|\dot{\Delta}_j u_{\tilde{\psi}_j(n)}\|_{L^{p_k}} = \liminf_{n\to\infty}\|\dot{\Delta}_j u_n\|_{L^{p_k}}<\infty$

$\tilde{u}_j\in L^{p_k}$, and in particular, in $S'(\mathbb{R}^d)$. Furthermore, $\tilde{u}_j\in C^\infty$ since $\operatorname{supp}\dot{\Delta}_j u_{\tilde{\psi}_j(n)}\subseteq K$ compact and $\dot{\Delta}_j u_{\tilde{\psi}_j(n)}\in C^\infty(\mathbb{R}^d)$ since it is convolution of Schwartz function with tempered distribution. Now, by Cantor's diagonalization process, $\exists$ $\psi(n)$ s.t. $\langle\dot{\Delta}_j u_{\psi(n)},\phi\rangle\longrightarrow\langle\tilde{u}_j,\phi\rangle$ as $n\to\infty$ and

$\|\tilde{u}_j\|_{L^{p_k}}\le\liminf_{n\to\infty}\|\dot{\Delta}_j u_n\|_{L^{p_k}}$.

Now $\left(2^{js_k}\|\dot{\Delta}_j u_{\psi(n)}\|_{L^{p_k}}\right)_{j\in\mathbb{Z}}$ bounded in $\ell^{r_k}(\mathbb{Z})$

Hence $\exists\ (\tilde{c}_j^k)_{j\in\mathbb{Z}}$ of ℓ^{r_k} s.t. $\forall\ (d_j)_{j\in\mathbb{Z}}$ with only finitely many $d_j \neq 0$.

$\lim_{n\to\infty} \sum_{j\in\mathbb{Z}} 2^{js_k} \| \dot{\Delta}_j u_{\psi(n)} \|_{L^{p_k}} d_j = \sum_{j\in\mathbb{Z}} \tilde{c}_j^k d_j$ and $\| (\tilde{c}_j^k)_j \|_{\ell^{r_k}} \leq \liminf_{n\to\infty} \| u_{\psi(n)} \|_{\dot{B}^{s_k}_{p_k, r_k}}$

Passing to the limit and using generalized Holder's Ineq gives us that $(2^{js_k} \| \tilde{u}_j \|_{L^{p_k}})_{j\in\mathbb{Z}} \in \ell^{r_k}$. Since $\mathcal{F}[(\tilde{u}_j)]$ supported in $2^j \mathcal{C}$,

by Lemma on page III, $\sum_{j\in\mathbb{Z}} \tilde{u}_j \to u \in \mathcal{S}_h'$ and so $\forall\ \phi \in \mathcal{S},\ M<N$

$$\left\langle \sum_{j=M}^{N} \dot{\Delta}_j u, \phi \right\rangle = \left\langle \sum_{j=M}^{N} \sum_{|j-j'|\leq 1} \dot{\Delta}_j \tilde{u}_{j'}, \phi \right\rangle. \text{ Hence,}$$

$$\sum_{j=M}^{N} \dot{\Delta}_j u = \lim_{n\to\infty} \sum_{j=M}^{N} \dot{\Delta}_j u_{\psi(n)} \quad \text{in } \mathcal{S}'(\mathbb{R}^d).$$ Since $s_1 < d_1/p_1$,

and $(u_{\psi(n)})_{n\in\mathbb{N}}$ bounded in $\dot{B}^{s_1}_{p_1,r_1}$, Bernstein's Ineq tells us that $\dot{S}_M u_{\psi(n)} \xrightarrow{M\to -\infty} 0$

uniformly. Similarly, $(\mathrm{Id} - \dot{S}_N) u_{\psi(n)} \to 0$ in $\dot{B}^{s_2-\gamma}_{p_2,r_2}$ uniformly

where γ picked so that $s_2 - \gamma < d_2/p_2$. Hence, u is indeed the limit of $u_{\psi(n)}$ in $\mathcal{S}'$. ∎

<u>Cor</u>: $\dot{B}^{s_1}_{p_1,r_1} \cap \dot{B}^{s_2}_{p_2,r_2}$ complete (with conditions from theorem above)

<u>Proof</u>: Consider Cauchy sequence $(u_n)_{n\in\mathbb{N}}$. Then it is bounded, so $\exists$ subsequence that converges to some $u \in \mathcal{S}'$. $\| u_{\psi(n)} - u_m \| + \| u_{\psi(n)} - u \| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} < \varepsilon$ ∎

V

Prop: If $p,r<\infty$ then $S_0(\mathbb{R}^d)=\{F\in S(\mathbb{R}^d) \mid 0\notin \operatorname{supp}\hat{F}\}$ is dense in $\dot{B}^s_{p,r}(\mathbb{R}^d)$

Proof: Say $u\in\dot{B}^s_{p,r}$. Since $r<\infty$, $\forall\, \varepsilon>0$, $\exists\, N\in\mathbb{N}$ s.t.

$$\|u-u_N\|_{\dot{B}^s_{p,r}}<\varepsilon \quad \text{with} \quad u_N=\sum_{|j|\le N}\dot{\Delta}_j u \in C^\infty \quad \text{since } \dot{\Delta}_j u\in C^\infty$$

Fix $\theta\in C_c(B(0,2))$ with $\theta\equiv 1$ on $B(0,1)$ $(0\le\theta\le 1)$. For $R>0$ set $\theta_R \overset{\text{def}}{=} \theta(\cdot/R)$ and fix M s.t. $M>N$. Define $u^R_{N,M}:=(Id-\dot{S}_{-M})(\theta_R u_N)$

Now, $\theta_R u_N\in C_c^\infty(\mathbb{R}^d)$ with $\operatorname{supp}(\theta_R u_N)\subseteq B(0,2R)$. Further, $\dot{S}_{-M}\theta_R u_N\in S(\mathbb{R}^d)$ since it is convolution of compactly supported distribution with Schwartz function. Finally, $\mathcal{F}(u^R_{N,M})=\mathcal{F}(\theta_R u_N)-\chi(2^M\xi)\mathcal{F}(\theta_R u_N)$.

Since $\operatorname{supp}\chi(2^M\xi)\subseteq 2^{-M}B(0,4/3)$ we get that $u^R_{N,M}\in S_0(\mathbb{R}^d)$.

Now, since $(Id-\dot{S}_{-M})u_N=u_N$ due to $\operatorname{supp}\hat{u}_N\cap\operatorname{supp}\hat{\dot{S}}_{-M}=\phi$, we get that $(Id-\dot{S}_{-M})((\theta_R-1)u_N)=u^R_{N,M}-u_N$.

Thus, by Bernstein's Ineq. $\forall\, j\in\mathbb{N}$, $k=\max(0,[s]+2)$

$$2^{js}\|\dot{\Delta}_j(u^R_{N,M}-u_N)\|_{L^p}\le 2^{-j}2^{jk}\|\dot{\Delta}_j(Id-\dot{S}_{-M})((\theta_R-1)u_N)\|_{L^p}$$
$$\le 2^{-j}\|\dot{\Delta}_j D^k(Id-\dot{S}_{-M})((\theta_R-1)u_N))\|_{L^p}$$
$$\lesssim 2^{-j}\|D^k[(Id-\dot{S}_{-M})((\theta_R-1)u_N)\|_{L^p}$$

If $-M-1\le j\le -1$ then $2^{js}\|\dot{\Delta}_j(u^R_{N,M}-u_N)\|_{L^p}\le C2^{js}\|(\theta_R-1)u_N\|_{L^p}$

If $j\le -M-2$, $\dot{\Delta}_j(u^R_{N,M}-u_N)=0$. Thus,

$$\|u^R_{N,M}-u_N\|_{\dot{B}^s_{p,r}}\lesssim\|D^k((\theta_R-1)u_N)\|_{L^p}+\left(\sum_{j=-M-1}^{-1}2^{jsr}\|(\theta_R-1)u_N\|^r_{L^p}\right)^{1/r}\longrightarrow 0 \text{ as } R\to\infty.$$

since $(\theta_R-1)u_N\to 0$ p.w a.e. and by Lebesgue Dominated Convergence Thm,

$|(\Theta_R - 1)u_N|^p \leq |u_N|^p$ and $u_N \in L^p$ since $\|u_N\|_{L^p} \leq \sum_{|j|\leq N} \|\dot{\Delta}_j u\|_{L^p} < \infty$.

Thus, $\exists$ $R > 0$ s.t. $\|u^R_{NM} - u_N\| < \varepsilon/2$. $\blacksquare$

<u>Prop</u>: For all $1 \leq p, r \leq \infty$ and $s \in \mathbb{R}$,
$$\begin{cases} \dot{B}^s_{p,r} \times \dot{B}^{-s}_{p',r'} \longrightarrow \mathbb{R} \\ \text{by} \quad (u, \phi) \longmapsto \sum_{|j-j'|\leq 1} \langle \Delta_j u, \Delta_{j'} \phi \rangle \end{cases}$$

defines a continuous linear functional on $\dot{B}^s_{p,r} \times \dot{B}^{-s}_{p',r'}$

If $u \in S'_h$ then $\|u\|_{\dot{B}^s_{p,r}} \leq C \sup_{\phi \in Q^{-s}_{p',r'}} \langle u, \phi \rangle$ where

$$Q^{-s}_{p',r'} = \{ \phi \in S(\mathbb{R}^d) \mid \|\phi\|_{\dot{B}^{-s}_{p',r'}} \leq 1 \}$$

<u>Proof (Sketch)</u>: $|\langle \dot{\Delta}_j u, \dot{\Delta}_{j'} \phi \rangle| \leq 2^{|s|} 2^{js} \|\dot{\Delta}_j u\|_{L^p} 2^{-j's} \|\dot{\Delta}_{j'} \phi\|_{L^{p'}}$

by Holder's Ineq. for $|j-j'| \leq 1$. Thus by Holder's ineq again,

$|\langle u, \phi \rangle| \lesssim \|u\|_{\dot{B}^s_{p,r}} \|\phi\|_{\dot{B}^{-s}_{p',r'}}$. So first part proven. Moving on,

$$\|u\|_{\dot{B}^s_{p,r}} = \sup_{N \in \mathbb{N}} \| (1_{|j|\leq N} 2^{js} \|\dot{\Delta}_j u\|_{L^p} \|_{\ell^r}$$

$$= \sup_{N \in \mathbb{N}} \sup_{(\alpha_j) \in Q^{r'}_N} \sum_{|j|\leq N} \|\dot{\Delta}_j u\|_{L^p} 2^{js} \alpha_j$$

Thus use $\|\dot{\Delta}_j u\|_{L^p} = \sup_{\substack{\phi \in S \\ \|\phi\|_{L^{p'}} \leq 1}} \langle \dot{\Delta}_j u, \phi \rangle$ $\therefore$ Define $\phi_N = \sum_{|j|\leq N} \alpha_j 2^{js} \dot{\Delta}_j \phi_j$

Then, $\|\phi_N\|_{\dot{B}^{-s}_{p',r'}} \leq C_{-s} \| 2^{js} \|\dot{\Delta}_j \phi_j\|_{L^p} \|_{\ell^r} < \infty$ (see Lemma on III)

Finally, $\|u\|_{\dot{B}^s_{p,r}} \leq \lim_{N\to\infty} \langle u, \phi_N \rangle + \varepsilon_N$ $\blacksquare$

VI

Prop: $s<0$, $1 \le p, r \le \infty$ and $u \in S_h'$. Then $u \in \dot{B}^s_{p,r}$ iff $2^{js}\|\dot{S}_j u\|_{L^p} \in \ell^r$

Proof: $2^{js}\|\dot{\Delta}_j u\|_{L^p} = 2^{js}\left(\|\dot{S}_{j+1}u - \dot{S}_j u\|_{L^p}\right) \le 2^{-s} 2^{(j+1)s}\|\dot{S}_{j+1}u\|_{L^p} + 2^{js}\|\dot{S}_j u\|_{L^p}$

$$\Rightarrow \|u\|_{\dot{B}^s_{p,r}} \le 2^{-s+1}\| 2^{js}\|\dot{S}_j u\|_{L^p}\|_{\ell^r}$$

Conversely, $\|2^{js}\|\dot{S}_j u\|_{L^p}\|_{\ell^r} \le \| 2^{js} \sum_{j' \le j-1} \|\dot{\Delta}_{j'} u\|_{L^p}\|_{\ell^r}$

$$= \|\sum_{j' \in \mathbb{Z}} \mathbb{1}_{[1,\infty)}(j-j')\, 2^{(j-j')s}\, 2^{j's}\|\dot{\Delta}_{j'} u\|_{L^p}\|_{\ell^r}$$

$$\le \|\mathbb{1}_{[1,\infty)} 2^{js}\|_{\ell^1}\, \|u\|_{\dot{B}^s_{p,r}} \lesssim \|u\|_{\dot{B}^s_{p,r}}.$$

$\boxtimes$

Some Alternate Characterizations of Besov Spaces for $u \in S_h'$

1): Say $s>0$, $(p,r) \in [1,\infty]^2$. Then $\|u\|_{\dot{B}^s_{p,r}} \approx \|\, \|t^{-s/2} e^{t\Delta} u\|_{L^p} \|_{L^r(\mathbb{R}^+, \frac{dt}{t})}$

2) $s \in (0,1)$, $(p,r) \in [1,\infty]^2$ Then $\|u\|_{\dot{B}^s_{p,r}} \approx \left\| \dfrac{\|\tau_{-y} u - u\|_{L^p}}{|y|^s} \right\|_{L^r(\mathbb{R}^d, \frac{dy}{|y|^d})}$

Embedding Theorems

Prop: $(p,q) \in [1,\infty]^2$, $p \le q$. Then:

1) $\dot{B}^{\frac{d}{p}-\frac{d}{q}}_{p,1} \xrightarrow{\text{continuous}} L^q$

2) $\dot{B}^{d/p}_{p,1} \xrightarrow{\text{cont.}} C_0$ (continuous vanishes at ∞) for $p<\infty$

3) $L^q \xrightarrow{\text{cont.}} \dot{B}^0_{q,\infty}$

Proof: 1) $u \in \dot{B}^{\frac{d}{p}-\frac{d}{q}}_{p,1}$, $u \in S_h'$ $\Rightarrow$ $u = \sum_j \dot{\Delta}_j u$, $\|\dot{\Delta}_j u\|_{L^q} \lesssim 2^{j(\frac{d}{p}-\frac{d}{q})}\|\dot{\Delta}_j u\|_{L^p}$

$$\Rightarrow \|u\|_{L^q} \le \sum_{j \in \mathbb{Z}} 2^{j(\frac{d}{p}-\frac{d}{q})}\|\dot{\Delta}_j u\|_{L^p} < \infty \text{ since } u \in \dot{B}^{\frac{d}{p}-\frac{d}{q}}_{p,1}$$

2) $p < \infty$ so S_0 dense in $\dot{B}^{d/p}_{p,1}$. Thus,

Paradifferential Calculus: Showing for $s < \frac{d}{p}$, $L^\infty \cap \dot{B}^s_{p,r}$ is an algebra.

Goal: Make $\dot{B}^s_{p,r}$ as close to an algebra as possible.

Construction! $uv = \dot{T}_u v + \dot{T}_v u + \dot{R}(u,v)$ where $\dot{T}_u v = \sum_{j \in \mathbb{Z}} \dot{S}_{j-1} u \, \dot{\Delta}_j v$

$$\dot{R}(u,v) = \sum_{|j-j'| \le 1} \dot{\Delta}_j u \, \dot{\Delta}_{j'} v$$

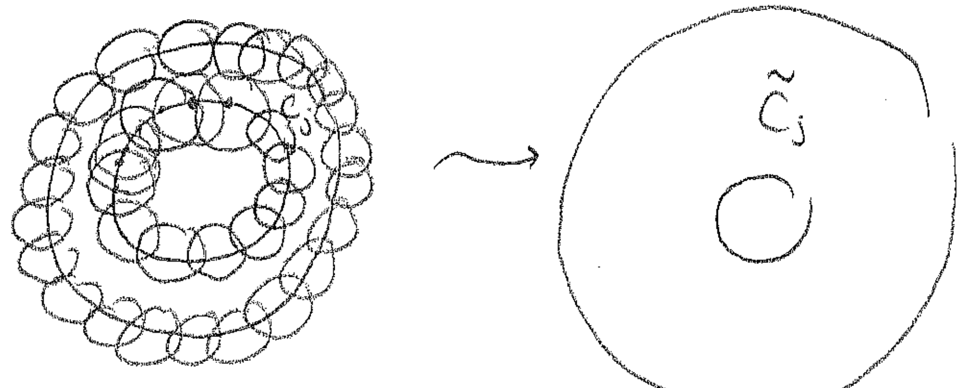
Let us assume to begin that these sums converge.

Prop: $\|\dot{T}_u v\|_{\dot{B}^s_{p,r}} \lesssim \|u\|_{L^\infty} \|v\|_{\dot{B}^s_{p,r}}$

Proof: $\mathcal{F}[\dot{S}_{j-1} u \, \dot{\Delta}_j v] = \underbrace{\chi(2^{-j+1}\xi)\hat{u}(\xi)}_{\substack{\text{supported in} \\ B_{j-1} = B(0,\, 2^{j-1}\frac{4}{3})}} * \underbrace{\varphi(2^{-j}\xi)\hat{v}(\xi)}_{\substack{\text{supported in} \\ C_j = \{2^j 3/4 \le |\xi| \le 8/3\, 2^j\}}}$

Thus, $\mathcal{F}[\dot{S}_{j-1} u \, \dot{\Delta}_j v]$ supported in $B_{j-1} + C_j = \tilde{C}_j$ annulus.

(imagine picture



where C_j thickened by balls $B(x, 2^{j-1}\frac{4}{3}) = B(x, 2^j \frac{2}{3})$ for all $x \in C_j$)

Thus, $\exists$ N s.t. (independent of j) $\dot{\Delta}_{j'}(\dot{S}_{j-1} u \, \dot{\Delta}_j v) = 0$ for $|j'-j| \ge N/2$.

$$\Rightarrow \|\dot{T}_u v\|_{\dot{B}^s_{p,r}} = \Big\| 2^{js} \| \dot{\Delta}_{j'} \big(\sum_{j \in \mathbb{Z}} \dot{S}_{j-1} u \, \dot{\Delta}_j v \big) \|_{L^p} \Big\|_{\ell^r_{j'}}$$

$$= \Big\| 2^{js} \Big\| \sum_{|j-j'| < N/2} \dot{\Delta}_{j'} (\dot{S}_{j-1} u \, \dot{\Delta}_j v) \Big\|_{L^p} \Big\|_{\ell^r_{j'}}$$

since $\dot{\Delta}_{j'} : L^p \to L^p$ bounded indep. of j' $\lesssim N \big\| 2^{js} \| \dot{S}_{j-1} u \, \dot{\Delta}_j v \|_{L^p} \big\|_{\ell^r} \lesssim \big\| 2^{js} \| \dot{S}_{j-1} u \|_{L^\infty} \| \dot{\Delta}_j v \|_{L^p} \big\|_{\ell^r}$ j' relabeled and $\dot{S}_{j-1} : L^\infty \longrightarrow L^\infty$ so $\|\dot{T}_u v\|_{\dot{B}^s_{p,r}} \lesssim \|u\|_{L^\infty} \|v\|_{\dot{B}^s_{p,r}}$

$\boxtimes$

<u>Prop</u>: $\|\dot{R}(u,v)\|_{\dot{B}^{s_1+s_2}_{p,r}} \lesssim \|u\|_{\dot{B}^{s_1}_{p_1,r_1}} \|v\|_{\dot{B}^{s_2}_{p_2,r_2}}$ for $s_1+s_2 > 0$ $(s_1,s_2) \in \mathbb{R}^2$

and $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} \leq 1$ $p_i, r_i \in [1,\infty]$

and $\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2} \leq 1$

<u>Proof</u>: $\dot{R}(u,v) = \sum_{j \in \mathbb{Z}} R_j$ with $R_j = \sum_{|\nu| \leq 1} \dot{\Delta}_{j-\nu} u \, \dot{\Delta}_j v$.

It can be checked that $\operatorname{supp} R_j \subseteq 2^j \tilde{B}$ for some ball $\tilde{B}$. Thus $\exists M$ s.t. $j' > j + M \Rightarrow \dot{\Delta}_{j'} R_j = 0$. Hence $\dot{\Delta}_{j'} \dot{R}(u,v) = \sum_{j \geq j'-M} \dot{\Delta}_{j'} R_j$

$$2^{j'(s_1+s_2)} \|\dot{\Delta}_{j'} \dot{R}(u,v)\|_{L^p} \lesssim C 2^{j'(s_1+s_2)} \sum_{\substack{|\nu| \leq 1 \\ j \geq j'-M}} \|\dot{\Delta}_{j-\nu} u \, \dot{\Delta}_j v\|_{L^p}$$

$$\lesssim 2^{j'(s_1+s_2)} \sum_{\substack{|\nu| \leq 1 \\ j \geq j'-M}} \|\dot{\Delta}_{j-\nu} u\|_{L^{p_1}} \|\dot{\Delta}_j v\|_{L^{p_2}}$$

$$\lesssim \sum_{\substack{|\nu| \leq 1 \\ j \geq j'-N_0}} 2^{-(j-j')(s_1+s_2)} 2^{(j-\nu)s_1} \|\dot{\Delta}_{j-\nu} u\|_{L^{p_1}} 2^{js_2} \|\dot{\Delta}_j v\|_{L^{p_2}}$$

$$= \left[2^{-j(s_1+s_2)} 1_{(-N_0,\infty)} \right] * \left[2^{(j-\nu)s_1} \|\dot{\Delta}_{j-\nu} u\|_{L^{p_1}} 2^{js_2} \|\dot{\Delta}_j v\|_{L^{p_2}} \right]$$

$$\Rightarrow \|\dot{R}(u,v)\|_{\dot{B}^s_{p,r}} \lesssim \left\| 2^{(j-\nu)s_1} \|\dot{\Delta}_{j-\nu} u\|_{L^{p_1}} 2^{js_2} \|\dot{\Delta}_j v\|_{L^{p_2}} \right\|_{\ell^r}$$

$$\lesssim \left\| 2^{(j-\nu)s_1} \|\dot{\Delta}_{j-\nu} u\|_{L^{p_1}} \right\|_{\ell^{r_1}} \left\| 2^{js_2} \|\dot{\Delta}_j v\|_{L^{p_2}} \right\|_{\ell^{r_2}}$$

$$\lesssim \|u\|_{\dot{B}^{s_1}_{p_1,r_1}} \|v\|_{\dot{B}^{s_2}_{p_2,r_2}}$$

Say $s < d/p$ or $s = d/p$ and $r=1$.

Prop: $\sum_{j\in\mathbb{Z}} \dot{S}_{j-1}u\dot{\Delta}_j v$ converges to $\dot{T}_u v \in \mathcal{S}_h'$ and $\|\dot{T}_u v\|_{\dot{B}^s_{p,r}} \lesssim \big\| 2^{js} \|\dot{S}_{j-1}u\dot{\Delta}_j v\|_{L^p} \big\|_{\ell^r}$

Proof: Remark after Lemma (III) (Lemma 2.23 in Bahouri) garuntees convergence in $\mathcal{S}_h'$ for conditions on s,p,r. Now, we just have to show that $\big\| 2^{js} \|\dot{S}_{j-1}u\,\dot{\Delta}_j v\|_{L^p}\big\|_{\ell^r} < \infty$. This is not so bad since $u \in L^\infty$, $v \in \dot{B}^s_{p,r}$

so $\big\| 2^{js} \|\dot{S}_{j-1}u\,\dot{\Delta}_j v\|_{L^p}\big\|_{\ell^r} \lesssim \|u\|_{L^\infty} \big\| 2^{js} \|\Delta_j v\|_{L^p}\big\|_{\ell^r} < \infty$.

Assume $s > 0$

Lemma: Say $(u_j)_{j\in\mathbb{Z}}$ sequence of smooth functions s.t. $\operatorname{supp}\hat{u}_j \subset 2^j B$ (B ball) and $\big\| 2^{js}\|u_j\|_{L^p}\big\|_{\ell^r} < \infty$. Then assuming $\sum_{j\in\mathbb{Z}} u_j$ converges to some $u \in \mathcal{S}_h'$ (which it does given conditions on s,p,r), we have that $\|u\|_{\dot{B}^s_{p,r}} \lesssim \big\| 2^{js}\|u_j\|_{L^p}\big\|_{\ell^r} < \infty$ so $u \in \dot{B}^s_{p,r}$

Proof: Similar to Lemma (III), we do

$$\|u\|_{\dot{B}^s_{p,r}} = \Big\| 2^{j's} \Big\| \dot{\Delta}_{j'} \sum_{j\in\mathbb{Z}} u_j \Big\|_{L^p} \Big\|_{\ell^r_{j'}} = \Big\| 2^{j's} \Big\| \sum_{j > j'+N} \Delta_{j'} u_j \Big\|_{L^p} \Big\|_{\ell^r_{j'}}$$

$$\leq \Big\| \sum_j 2^{(j'-j)s} \mathbb{1}_{(-\infty,-N]}(j'-j) 2^{js} \underbrace{\|\Delta_{j'} u_j\|_{L^p}}_{\lesssim \|u_j\|_{L^p}} \Big\|_{\ell^r_{j'}} \lesssim \big\| 2^{js}\|u_j\|_{L^p}\big\|_{\ell^r}$$

Thm: For $s>0$, $p,r\in[1,\infty]$ satisfying ($s=\frac{d}{p}$ and $r=1$) or ($s<d/p$), $L^\infty \cap \dot{B}^s_{p,r}$ is an algebra.

Proof: $\|uv\|_{\dot{B}^s_{p,r}} \leq \|\dot{T}_u v\|_{\dot{B}^s_{p,r}} + \|\dot{T}_v u\|_{\dot{B}^s_{p,r}} + \|\dot{R}(u,v)\|_{\dot{B}^s_{p,r}}$

$$\Rightarrow \|uv\|_{\dot{B}^s_{p,r}} \lesssim \|u\|_{L^\infty}\|v\|_{\dot{B}^s_{p,r}} + \|u\|_{\dot{B}^s_{p,r}}\|v\|_{L^\infty} + \underbrace{\|u\|_{\dot{B}^0_{\infty,\infty}}}_{\lesssim \|u\|_{L^\infty}}\|v\|_{\dot{B}^s_{p,r}} < \infty.$$

$$\Rightarrow uv \in \dot{B}^s_{p,r} \cap L^\infty$$

$\blacksquare$