MATH 361 — HOMEWORK 7.

due on Friday, October 30.

by J. E. Marsden and M. J. Hoffman

Topics:
• 5. Uniform Convergence
  – 5.7 The Contraction Mapping Principle and Its Applications
  – 5.8 The Stone-Weierstrass Theorem
  – 5.10 Power Series

Seventh Homework Assignment.

Reading:
• Read Read Section 5.10. Read your notes. Summation by parts is
  in the Proofs section for Section 5.9.

Exercises:

Problem 1. Prove that if the series $\sum_{n=0}^{\infty} a_n$ is convergent, then

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} k a_k = 0.$$ 

(Hint: Use summation by parts. This is a result of Kronecker.)

Problem 2. Prove problem 5 on page 294 by using the expansion

$$-\ln(1 - z) = \frac{z}{1} + \frac{z^2}{2} + \cdots + \frac{z^n}{n} + \cdots$$

proven in class, then $\ln(1 + z) - \ln(1 - z)$ and finally using Abel’s Theorem
for $z_0 = i$.

Problem 3. Suppose $(E, \| \|)$ and $(F, \| \|)$ are normed vector spaces, and
$T \in \mathcal{L}(E, F)$. Show that if for every sequence $\{x_n\}_{n \in \mathbb{N}}, x_n \in E$ such that
$\lim_{n} x_n = 0$ it follows that the sequence $\{T(x_n)\}_{n \in \mathbb{N}}$ is bounded (in $F$), then
$T$ is continuous (i.e. $T \in L(E, F)$).

Problem 4. Suppose $(E, \| \|)$ and $(F, \| \|)$ are real normed vector spaces, and
$u : E \to F$ a map satisfying

$$u(x + y) = u(x) + u(y), \quad \text{for every } x, y \in E.$$ 

Show that if $u$ is bounded on the unit disc $D_1(0)$, then $u \in L(E, F)$.
(Hint. Show first that $u(rx) = ru(x)$, for every rational number $r \in \mathbb{Q}$.)
Problems:
  • Page 316: problems: 62, 63, 66