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Topics:
• Review of Math 360
• 5. Uniform Convergence
  – 5.1 Pointwise and Uniform Convergence
  – 5.2 The Weierstrass M Test
  – 5.5 The Space of Continuous Functions
  – 5.6 The Arzela-Ascoli Theorem
  – 5.7 The Contraction Mapping Principle and Its Applications

Fourth Homework Assignment.

Reading:
• Read section 5.7 of Chapter 5, paying close attention to the examples. Read your notes.

Exercises:

Problem 1. Write down the formula for the Euler Approximation of the solution to the initial value problem
\[
\frac{dx}{dt} = F(t, x) \\
x(t_0) = x_0
\]
for values \( t < t_0 \).

Problem 2. Prove that if \((N_i, \rho_i), i = 1, 2\), are metric spaces and \( F : N_1 \to N_2 \) is a uniformly continuous function that maps bounded sets into bounded sets, then the map:
\[
F_b(S, N_1) \ni f \mapsto F \circ f \in F_b(S, N_2)
\]
is uniformly continuous. (Here \( S \) is any nonempty set, and \( F_b(S, N_i) \) is a metric space with the metric \( \rho_{b,\infty} \).

Problem 3. Prove that if \((N_i, \rho_i), i = 1, 2\), are metric spaces and \( F : N_1 \to N_2 \) is a continuous function, then the map:
\[
C(K, N_1) \ni f \mapsto F \circ f \in C(K, N_2)
\]
is continuous, for every compact metric space \((K,d)\).

**Problem 4.** (Dini’s Theorem) Let \((K,d)\) be a compact metric space and \(\{f_k\}_k\) a sequence of continuous functions
\[
f_k : K \rightarrow \mathbb{R},
\]
such that
(a) \(f_k(x) \geq 0\), for every \(x \in K\),
(b) \(f_k(x) \leq f_\ell(x)\) for every \(x \in K\) and every \(k \geq \ell\).

Prove that if \(\{f_k\}_k\) converges pointwise to zero then it converges uniformly to zero.

**Problem 5.** For what intervals \([0,r]\), \(r \leq 1\), is \(f : [0,r] \rightarrow [0,r], f(x) = x^2\), a contraction, a strict contraction and a uniform strict contraction respectively?

How about \(f(x) = x^a\), \(a \geq 0\)?

(Recall that \(T : M \rightarrow M\) is a:
- contraction if \(d(T(x),T(y)) \leq d(x,y)\), for every \(x, y \in M\),
- strict contraction if \(d(T(x),T(y)) < d(x,y)\), for every \(x, y \in M, x \neq y\),
- uniform strict contraction if there exists \(c \in [0,1)\) such that \(d(T(x),T(y)) \leq cd(x,y)\), for every \(x, y \in M\).

The Contraction Mapping Principle is really about uniform strict contractions.)

**Problem 6.** Suppose \(f\) is a solution to the initial value problem
\[
\frac{dx}{dt} = x^2 + t^2
\]
\(x(0) = 1\)

What is the third derivative of \(f\) at \(t = 0\)?

**Problems:**
- Page 283: problems 5, 7
- Page 316: problems 25, 27.