Integrating Factors and Reduction of Order

Math 240 — Calculus III

Summer 2015, Session II

Monday, August 3, 2015

Agenda

1. Integrating factors

2. Reduction of order

Introduction

The reduction of order technique, which applies to second-order linear differential equations, allows us to go beyond equations with constant coefficients, provided that we already know one solution.

If our differential equation is

$$y'' + a_1(x)y' + a_2(x)y = F(x),$$

and we know the solution, $y_1(x)$, to the associated homogeneous equation, this method will furnish us with another, independent solution.

To accomplish the process, we will make use of *integrating factors*.

Integrating factors

Integrating factors are a technique for solving first-order linear differential equations, that is, equations of the form

$$a(x)\frac{dy}{dx} + b(x)y = r(x).$$

Assuming $a(x) \neq 0$, we can divide by $a(x)$ to put the equation in **standard form**:

$$\frac{dy}{dx} + p(x)y = q(x).$$

The main idea is that the left-hand side looks almost like the result of the product rule for derivatives. If $I(x)$ is another function then

$$\frac{d}{dx}(Iy) = I\frac{dy}{dx} + \frac{dI}{dx}y.$$

The standard form equation is missing an I in front of $\frac{dy}{dx}$, so let's multiply it by I.

Integrating factors

When we multiply our equation by I, we get

$$I\frac{dy}{dx} + Ip(x)y = Iq(x),$$

so in order for the left-hand side to be $\frac{d}{dx}(Iy)$, we need to have

$$\frac{dI}{dx} = p(x)I.$$

Rearranging this into

$$\frac{dI}{I} = p(x)\,dx,$$

we can solve:

$$I(x) = c_1 e^{\int p(x)\,dx}.$$

Since we only need one function I, let's set $c_1 = 1$.

Integrating factors

Using this I, we rewrite our equation as

$$\frac{d}{dx}(Iy) = q(x)I,$$

then integrate and divide by I to get

$$y(x) = \frac{1}{I}\left(\int q(x)I\,dx + c\right).$$

Our I is called an **integrating factor** because it is something we can multiply by (a factor) that allows us to integrate.

Example

Find a solution to

$$y' + xy = xe^{x^2/2}.$$

1. Find the integrating factor
$$I(x) = e^{\int x\,dx} = e^{x^2/2}.$$
2. Multiply it into the original equation:
$$\frac{d}{dx}\left(e^{x^2/2}y\right) = e^{x^2/2}y' + xe^{x^2/2}y = xe^{x^2}.$$
3. Integrate both sides:
$$e^{x^2/2}y = \frac{1}{2}e^{x^2} + c.$$
4. Divide by I to find the solution
$$y(x) = e^{-x^2/2}\left(\frac{1}{2}e^{x^2} + c\right).$$

Example

Solve, for $x > 0$, the equation

$$xy' + 2y = \cos x.$$

1. Write the equation in standard form:
$$y' + \frac{2}{x}y = \frac{\cos x}{x}.$$
2. An integrating factor is
$$I(x) = e^{2\ln x} = x^2.$$
3. Multiply by I to get
$$\frac{d}{dx}(x^2 y) = x\cos x.$$
4. Integrate and divide by x^2 to get
$$y(x) = \frac{x\sin x + \cos x + c}{x^2}.$$

Reduction of order

We now turn to second-order equations

$$y'' + a_1(x)y' + a_2(x)y = F(x).$$

We know that the general solution to such an equation will look like

$$y(x) = c_1y_1(x) + c_2y_2(x) + y_p(x).$$

Suppose that we know $y_1(x)$. We will guess the solution $y(x) = u(x)y_1(x)$. Plugging it into our original equation yields

$$u''y_1 + u'\left(2y_1' + a_1(x)y_1\right) = F(x).$$

If we let $w = u'$ then we have reduced our second-order equation to the first-order

$$w' + \left(\frac{2y_1'}{y_1} + a_1\right)w = \frac{F(x)}{y_1}.$$

Reduction of order

We may solve

$$w' + \left(\frac{2y_1'}{y_1} + a_1\right) w = \frac{F(x)}{y_1}$$

using the integrating factor technique:

$$I(x) = y_1^2(x) e^{\int^x a_1(s)\, ds}$$

and

$$w(x) = \frac{1}{I(x)} \int^x \frac{I(s)F(s)}{y_1(s)}\, ds + \frac{c_1}{I(x)}.$$

Then integrate w to find u:

$$u(x) = \int^x \frac{1}{I(t)} \int^t \frac{I(s)F(s)}{y_1(s)}\, ds\, dt + c_1 \int^x \frac{1}{I(s)}\, ds + c_2.$$

Reduction of order

Finally, we get

$$y(x) = u(x)y_1(x) = c_1 y_1(x) \int^x \frac{1}{I(s)}\,ds + c_2 y_1(x) + y_1(x) \int^x \frac{1}{I(t)} \int^t \frac{I(s)F(s)}{y_1(s)}\,ds\,dt.$$

Using $F = 0$ gives us the two fundamental solutions

$$y(x) = y_1(x) \text{ and } y(x) = y_1(x) \int^x \frac{1}{I(s)}\,ds.$$

And using $c_1 = c_2 = 0$, we get a particular solution

$$y_p(x) = y_1(x) \int^x \frac{1}{I(t)} \int^t \frac{I(s)F(s)}{y_1(s)}\,ds\,dt.$$

Example

Determine the general solution to

$$xy'' - 2y' + (2 - x)y = 0, \quad x > 0,$$

given that one solution is $y_1(x) = e^x$.

1. Set up the equation for w:
$$w' + \frac{2(x-1)}{x} w = 0.$$
2. Solve for w:
$$w(x) = c_1 x^2 e^{-2x}.$$
3. Integrate to find
$$u(x) = \int w(x)\,dx + c_2 = -\frac{1}{4} c_1 e^{-2x}(1 + 2x + 2x^2) + c_2.$$
4. Multiply by y_1 for the general solution:
$$y(x) = c_1 e^{-x}(1 + 2x + 2x^2) + c_2 e^x.$$

Example

Determine the general solution to

$$x^2y'' + 3xy' + y = 4\ln x, \quad x > 0,$$

by first finding solutions to the associated homogeneous equation of the form $y(x) = x^r$.

1. Find $y_1(x) = x^{-1}$.
2. Put the equation in standard form by dividing by x^2:
$$y'' + 3x^{-1}y' + x^{-2}y = 4x^{-2}\ln x.$$
3. Set up the equation $w' + x^{-1}w = 4x^{-1}\ln x$.
4. Find $w(x) = 4(\ln x - 1) + c_1x^{-1}$.
5. Then $u(x) = 4x(\ln x - 2) + c_1\ln x + c_2$.
6. Multiply by $y_1(x) = x^{-1}$:
$$y(x) = 4(\ln x - 2) + c_1x^{-1}\ln x + c_2x^{-1}.$$