Math 240

Definition

Computation and Properties

Chains

Jordan canonical form

Generalized Eigenvectors

Math 240 — Calculus III

Summer 2015, Session II

Thursday, July 23, 2015



Math 240

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Chains

Jordan canonical form

1. Definition

2. Computation and Properties

Agenda

3. Chains

4. Jordan canonical form



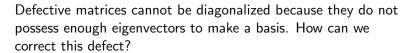
Math 240

Definition

Computation and Properties

Chains

Jordan canonical form



Motivation

Example The matrix $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ is defective. 1. Only eigenvalue is $\lambda = 1$.

- 2. $A I = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$
- 3. Single eigenvector $\mathbf{v} = (1, 0)$.
- 4. We could use $\mathbf{u}=(0,1)$ to complete a basis.

5. Notice that $(A - I)\mathbf{u} = \mathbf{v}$ and $(A - I)^2\mathbf{u} = \mathbf{0}$.

Maybe we just didn't multiply by $A - \lambda I$ enough times.



Math 240

Definition

Computation and Properties

Chains

Jordan canonical form

Definition

If A is an $n \times n$ matrix, a **generalized eigenvector** of A corresponding to the eigenvalue λ is a nonzero vector **x** satisfying

$$(A - \lambda I)^p \mathbf{x} = \mathbf{0}$$

Definition

for some positive integer p. Equivalently, it is a nonzero element of the nullspace of $(A - \lambda I)^p$.

Example

- Eigenvectors are generalized eigenvectors with p = 1.
- ▶ In the previous example we saw that $\mathbf{v}=(1,0)$ and $\mathbf{u}=(0,1)$ are generalized eigenvectors for

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \text{ and } \lambda = 1.$$



Math 240

Definition

Computation and Properties

Chains

Jordan canonical form



Example

Determine generalized eigenvectors for the matrix

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{bmatrix}.$$

- 1. Characteristic polynomial is $(3 \lambda)(1 \lambda)^2$.
- 2. Eigenvalues are $\lambda = 1, 3$.
- 3. Eigenvectors are

$$\begin{aligned} \lambda_1 &= 3: & \mathbf{v}_1 &= (1,2,2), \\ \lambda_2 &= 1: & \mathbf{v}_2 &= (1,0,0). \end{aligned}$$

4. Final generalized eigenvector will a vector $\mathbf{v}_3 \neq \mathbf{0}$ such that

$$(A - \lambda_2 I)^2 \mathbf{v}_3 = \mathbf{0} \text{ but } (A - \lambda_2 I) \mathbf{v}_3 \neq \mathbf{0}.$$

Pick $\mathbf{v}_3 = (0, 1, 0)$. Note that $(A - \lambda_2 I) \mathbf{v}_3 = \mathbf{v}_2.$



Math 240

Definition

Computation and Properties

Chains

Jordan canonical form

Facts about generalized eigenvectors

How many powers of $(A - \lambda I)$ do we need to compute in order to find all of the generalized eigenvectors for λ ?

Fact

If A is an $n \times n$ matrix and λ is an eigenvalue with algebraic multiplicity k, then the set of generalized eigenvectors for λ consists of the nonzero elements of nullspace $((A - \lambda I)^k)$. In other words, we need to take at most k powers of $A - \lambda I$ to find all of the generalized eigenvectors for λ .



Math 240

Definition

Computation and Properties

Chains

Jordan canonical form

Computing generalized eigenvectors

Example

Determine generalized eigenvectors for the matrix

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 1 & 2 \\ 0 & -1 & 1 \end{bmatrix}.$$

- 1. Single eigenvalue of $\lambda = 1$.
- 2. Single eigenvector $\mathbf{v}_1 = (-2, 0, 1)$.
- 3. Look at

$$(A - I)^2 = \begin{bmatrix} 2 & 0 & 4 \\ 0 & 0 & 0 \\ -1 & 0 & -2 \end{bmatrix}$$

to find generalized eigenvector $\mathbf{v}_2 = (0, 1, 0)$. 4. Finally, $(A - I)^3 = \mathbf{0}$, so we get $\mathbf{v}_3 = (1, 0, 0)$.



Math 240

Definition

Computation and Properties

Chains

Jordan canonical form

Facts about generalized eigenvectors

The aim of generalized eigenvectors was to enlarge a set of linearly independent eigenvectors to make a basis. Are there always enough generalized eigenvectors to do so?

Fact

If λ is an eigenvalue of A with algebraic multiplicity k, then nullity $((A - \lambda I)^k) = k$.

In other words, there are k linearly independent generalized eigenvectors for $\lambda.$

Corollary

If A is an $n \times n$ matrix, then there is a basis for \mathbb{R}^n consisting of generalized eigenvectors of A.



Math 240

Definition

Computation and Properties

Chains

Jordan canonical form

Computing generalized eigenvectors

Example

Determine generalized eigenvectors for the matrix

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 1 & 2 \\ 0 & -1 & 1 \end{bmatrix}.$$

- 1. From last time, we have eigenvalue $\lambda=1$ and eigenvector $\mathbf{v}_1=(-2,0,1).$
- 2. Solve $(A I)\mathbf{v}_2 = \mathbf{v}_1$ to get $\mathbf{v}_2 = (0, -1, 0)$.
- 3. Solve $(A I)\mathbf{v}_3 = \mathbf{v}_2$ to get $\mathbf{v}_3 = (-1, 0, 0)$.



Math 240

Definition

Computation and Properties

Chains

Jordan canonical form

Chains of generalized eigenvectors

Let A be an $n\times n$ matrix and ${\bf v}$ a generalized eigenvector of A corresponding to the eigenvalue $\lambda.$ This means that

$$(A - \lambda I)^p \mathbf{v} = \mathbf{0}$$

for a positive integer p.

If $0 \leq q < p$, then

$$(A - \lambda I)^{p-q} (A - \lambda I)^q \mathbf{v} = \mathbf{0}.$$

That is, $(A - \lambda I)^q \mathbf{v}$ is also a generalized eigenvector corresponding to λ for $q = 0, 1, \dots, p - 1$.

Definition

If p is the smallest positive integer such that $(A-\lambda I)^p\, {\bf v}={\bf 0},$ then the sequence

$$(A - \lambda I)^{p-1} \mathbf{v}, \ (A - \lambda I)^{p-2} \mathbf{v}, \ \dots, \ (A - \lambda I) \mathbf{v}, \ \mathbf{v}$$



is called a **chain** or **cycle** of generalized eigenvectors. The integer p is called the **length** of the cycle.

Math 240

Definition

Computation and Properties

Chains

Jordan canonical form

Chains of generalized eigenvectors

Example

In the previous example,

$$A - \lambda I = \begin{bmatrix} 0 & 2 & 0 \\ 1 & 0 & 2 \\ 0 & -1 & 0 \end{bmatrix}$$

and we found the chain

$$\mathbf{v} = \begin{bmatrix} -1\\0\\0 \end{bmatrix}, \ (A - \lambda I)\mathbf{v} = \begin{bmatrix} 0\\-1\\0 \end{bmatrix}, \ (A - \lambda I)^2\mathbf{v} = \begin{bmatrix} -2\\0\\1 \end{bmatrix}$$

Remark

The terminal vector in a chain is always an eigenvector.

Fact

The generalized eigenvectors in a chain are linearly independent.



Math 240

Definition

Computation and Properties

Chains

Jordan canonical form What's the analogue of diagonalization for defective matrices? That is, if $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ are the linearly independent generalized eigenvectors of A occurring in chains, what does the matrix $S^{-1}AS$ look like, where $S = [\mathbf{v}_1 \quad \mathbf{v}_2 \quad \cdots \quad \mathbf{v}_n]$?

Introduction to Jordan form

Suppose that $\mathbf{v}_1, \mathbf{v}_1, \dots, \mathbf{v}_k$ is a chain of generalized eigenvectors, so that $(A - \lambda I)\mathbf{v}_i = \mathbf{v}_{i-1}$ for i > 1 and $(A - \lambda I)\mathbf{v}_1 = \mathbf{0}$. Then we have

$$\begin{split} A\mathbf{v}_i &= \lambda \mathbf{v}_i + \mathbf{v}_{i-1} \text{ for } i > 1\\ \text{ and } A\mathbf{v}_1 &= \lambda \mathbf{v}_1. \end{split}$$



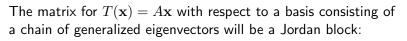
Math 240

Definition

Computation and Properties

Chains

Jordan canonical form



lordan blocks

Definition

If $\boldsymbol{\lambda}$ is a real number, then the square matrix of the form

$$J_{\lambda} = \begin{bmatrix} \lambda & 1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & 0 & \cdots & 0 \\ 0 & 0 & \lambda & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \cdots & \lambda & 1 \\ 0 & 0 & \cdots & \cdots & 0 & \lambda \end{bmatrix}$$

is called a Jordan block corresponding to $\lambda.$



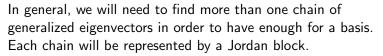
Math 240

Definition

Computation and Properties

Chains

Jordan canonical form



Jordan canonical form

Definition

A square matrix consisting of Jordan blocks centered along the main diagonal and zeros elsewhere is said to be in **Jordan** canonical form (JCF).

Theorem

If S is the matrix whose columns are a basis of generalized eigenvectors of A arranged in chains, then $S^{-1}AS$ is a matrix in JCF. It is unique up to a rearrangement of the Jordan blocks. We may therefore refer to this matrix as the Jordan canonical form of A, and we see that every matrix is similar to a matrix in JCF.



Math 240

Definition

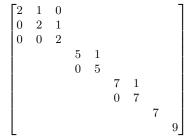
Computation and Properties

Chains

Jordan canonical form



The matrix



is in JCF. It contains five Jordan blocks.

- \blacktriangleright Any diagonal matrix is in JCF. All of its Jordan blocks are $1\times 1.$
- The matrix

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

is in JCF. It has two blocks of sizes 2 and 1.



Math 240

Definition

Computation and Properties

Chains

Jordan canonical form

Theorem

Two $n \times n$ matrices are similar if and only if they have the same Jordan canonical form (up to a rearrangement of the Jordan blocks).

Our main use for JCF will be solving $\mathbf{x}' = A\mathbf{x}$ when the matrix A is defective.

