Diagonalization

Math 240 — Calculus III

Summer 2015, Session II

Monday, July 20, 2015

Agenda

1. Change of Basis
2. Diagonalization
   - Uses for diagonalization

The change of basis matrix

Definition

Suppose V is a vector space with two bases

$$B = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\} \text{ and } C = \{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_n\}.$$

The **change of basis matrix from B to C** is the matrix $S = [s_{ij}]$, where

$$\mathbf{v}_j = s_{1j}\mathbf{w}_1 + s_{2j}\mathbf{w}_2 + \cdots + s_{nj}\mathbf{w}_n.$$

In other words, it is the matrix whose columns are the vectors of B expressed in coordinates via C.

Example

Consider the bases $B = \{1, 1+x, (1+x)^2\}$ and $C = \{1, x, x^2\}$ for P_2. The change of basis matrix from B to C is

$$S = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}.$$

Using the change of basis matrix

Theorem

Suppose V is a vector space with bases B and C, and S is the change of basis matrix from B to C. If $\mathbf{v}$ is a column vector of coordinates with respect to B, then $S\mathbf{v}$ is the column vector of coordinates for the same vector with respect to C.

The change of basis matrix turns B-coordinates into C-coordinates.

Example

Using the change of basis matrix from the previous slide, we can compute

$$(1+x)^2 - 2(1+x) = S\begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = x^2 - 1.$$

Inverse change of basis

Suppose we have bases B and C for the vector space V. There is a change of basis matrix S from B to C and also a change of basis matrix P from C to B. Then

$$PS\mathbf{e}_1 = \mathbf{e}_1, \quad PS\mathbf{e}_2 = \mathbf{e}_2, \quad \ldots, \quad PS\mathbf{e}_n = \mathbf{e}_n$$

and

$$SP\mathbf{e}_1 = \mathbf{e}_1, \quad SP\mathbf{e}_2 = \mathbf{e}_2, \quad \ldots, \quad SP\mathbf{e}_n = \mathbf{e}_n.$$

Theorem

In the notation above, S and P are inverse matrices.

Matrix representations for linear transformations

Theorem

Let $T : V \to W$ be a linear transformation and A a matrix representation for T relative to bases C for V and D for W. Suppose B is another basis for V and E is another basis for W, and let S be the change of basis matrix from B to C and P the change of basis matrix from E to D.

- *The matrix representation of T relative to B and D is AS.*
- *The matrix representation of T relative to C and E is $P^{-1}A$.*
- *The matrix representation of T relative to B and E is $P^{-1}AS$.*

$$\begin{tikzcd} V_C \arrow[r, "A"] & W_D \\ V_B \arrow[u, "S"] \arrow[ur, "AS"'] & \end{tikzcd} \qquad \begin{tikzcd} V_C \arrow[r, "A"] \arrow[dr, "P^{-1}A"'] & W_D \\ & W_E \arrow[u, "P"'] \end{tikzcd} \qquad \begin{tikzcd} V_C \arrow[r, "A"] & W_D \\ V_B \arrow[u, "S"] \arrow[r, "P^{-1}AS"] & W_E \arrow[u, "P"'] \end{tikzcd}$$

Similar matrices

For eigenvectors and diagonalization, we are interested in linear transformations $T : V \to V$.

Corollary

Let A be a matrix representation of a linear transformation $T : V \to V$ relative to the basis B. If S is the change of basis matrix from a basis C to B, then the matrix representation of T relative to C is $S^{-1}AS$.

Definition

Let A and B be $n \times n$ matrices. We say that A is **similar** to B if there is an invertible matrix S such that $B = S^{-1}AS$.

Similar matrices represent the same linear transformation relative to different bases.

Eigenvalues of similar matrices

Theorem

Similar matrices have the same eigenvalues (including multiplicities).

But,

the eigenvectors of similar matrices are different.

Proof.

If A is similar to B, then $B = S^{-1}AS$ for some invertible matrix S. Thus,

$$\begin{aligned} \det(B - \lambda I) &= \det\left(S^{-1}AS - \lambda S^{-1}S\right) \\ &= \det\left(S^{-1}(A - \lambda I)S\right) \\ &= \det\left(S^{-1}S\right)\det(A - \lambda I) = \det(A - \lambda I). \end{aligned}$$

$\mathcal{Q.E.D.}$

Diagonalization

Definition

The diagonal matrix with main diagonal $\lambda_1, \lambda_2, \ldots, \lambda_n$ is denoted

$$\operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n) = \begin{bmatrix} \lambda_1 & & & \text{\Huge 0} \\ & \lambda_2 & & \\ & & \ddots & \\ \text{\Huge 0} & & & \lambda_n \end{bmatrix}.$$

If A is a square matrix with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, the simplest matrix with those eigenvalues is $\operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$.

Definition

A square matrix that is similar to a diagonal matrix is called **diagonalizable**.

Our question is, which matrices are diagonalizable?

Diagonalization

Theorem

An $n \times n$ matrix A is diagonalizable if and only if it is nondefective. In this case, if $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ denote n linearly independent eigenvectors of A and

$$S = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \end{bmatrix},$$

then

$$S^{-1}AS = \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n),$$

where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of A (not necessarily distinct) corresponding to the eigenvectors $\mathbf{v}_1, \mathbf{v}_1, \ldots, \mathbf{v}_n$.

Example

Verify that

$$A = \begin{bmatrix} 3 & -2 & -2 \\ -3 & -2 & -6 \\ 3 & 6 & 10 \end{bmatrix}$$

is diagonalizable and find an invertible matrix S such that $S^{-1}AS$ is diagonal.

1. The characteristic polynomial of A is $-(\lambda-4)^2(\lambda-3)$.
2. The eigenvalues of A are $\lambda = 4, 4, 3$.
3. The corresponding eigenvectors are

$$\begin{aligned} \lambda = 4: &\qquad \mathbf{v}_1 = (-2, 0, 1), \qquad \mathbf{v}_2 = (-2, 1, 0), \\ \lambda = 3: &\qquad \mathbf{v}_3 = (1, 3, -3). \end{aligned}$$

4. A is nondefective, hence diagonalizable. Let $S = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{bmatrix}$.
5. Then, according to the theorem, we will have $S^{-1}AS = \operatorname{diag}(4, 4, 3)$.

Raising matrices to high powers

If A is a square matrix, you may want to compute A^k for some large number k. This might be a lot of work. Notice, however, that if $A = SDS^{-1}$, then

$$A^2 = (SDS^{-1})(SDS^{-1}) = SD(SS^{-1})DS^{-1} = SD^2S^{-1},$$
$$A^3 = A^2A = (SD^2S^{-1})(SDS^{-1}) = SD^3S^{-1},$$

etc.

We can compute D^k fairly easily by raising each entry to the k-th power.

Theorem

If A is a nondefective matrix and $A = SDS^{-1}$, then

$$A^k = SD^kS^{-1}.$$

The matrix exponential function

Recall the exponential power series

$$e^t = 1 + t + \frac{1}{2}t^2 + \frac{1}{6}t^3 + \cdots + \frac{1}{n!}t^n + \ldots.$$

Definition

If A is an $n \times n$ matrix, we define the **matrix exponential function** by

$$e^{At} = I_n + At + \frac{1}{2}(At)^2 + \frac{1}{6}(At)^3 + \cdots + \frac{1}{n!}(At)^n + \ldots.$$

It is an $n \times n$ matrix function.

Notice that if $A = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$, then $e^{At} = \operatorname{diag}(e^{\lambda_1 t}, \ldots, e^{\lambda_n t})$.

Theorem

If A is diagonalizable as $S^{-1}AS = D$, then $e^{At} = Se^{Dt}S^{-1}$.

Solving linear systems of differential equations

We saw last time that linear systems of differential equations with diagonal coefficient matrices have particularly simple solutions. Diagonalization allows us to turn a linear system with a nondefective coefficient matrix into such a diagonal system.

Theorem

Let $\mathbf{x}' = A\mathbf{x}$ be a homogeneous system of linear differential equations, for A an $n \times n$ matrix with real entries. If A is nondefective and $S^{-1}AS = \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$, then all solution to $\mathbf{x}' = A\mathbf{x}$ are given by

$$\mathbf{x} = S\mathbf{y}, \ \textit{where } \mathbf{y} = \begin{bmatrix} c_1 e^{\lambda_1 t} \\ c_2 e^{\lambda_2 t} \\ \vdots \\ c_n e^{\lambda_n t} \end{bmatrix},$$

where $c_1, c_2, \ldots, c_n$ are scalars.