Math 240

Linear Transformations

Transformation of Euclidean space

Kernel and Range

The matrix of a linear trans.

Composition of linear trans. Kernel and

Linear Transformations

Math 240 — Calculus III

Summer 2015, Session II

Wednesday, July 15, 2015



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Agenda

2. Kernel and Range

3. The matrix of a linear transformation Composition of linear transformations Kernel and Range



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In the $m\times n$ linear system

 $A\mathbf{x} = \mathbf{0},$

we can regard A as transforming elements of \mathbb{R}^n (as column vectors) into elements of \mathbb{R}^m via the rule

 $T(\mathbf{x}) = A\mathbf{x}.$

Then solving the system amounts to finding all of the vectors $\mathbf{x} \in \mathbb{R}^n$ such that $T(\mathbf{x}) = \mathbf{0}$.

Solving the differential equation

$$y'' + y = 0$$

is equivalent to finding functions y such that $T(y) = \mathbf{0},$ where T is defined as

$$T(y) = y'' + y.$$



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Definition

Let V and W be vector spaces with the same scalars. A mapping $T: V \to W$ is called a **linear transformation** from V to W if it satisfies

1.
$$T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$$
 and
2. $T(c \mathbf{v}) = c T(\mathbf{v})$

for all vectors $\mathbf{u}, \mathbf{v} \in V$ and all scalars c. The space V is called the **domain** and W the **codomain** of T.

Examples

- ▶ $T : \mathbb{R}^n \to \mathbb{R}^m$ defined by $T(\mathbf{x}) = A\mathbf{x}$, where A is an $m \times n$ matrix
- $\blacktriangleright \ T: C^k(I) \to C^{k-2}(I) \text{ defined by } T(y) = y'' + y$
- ▶ $T: M_{m \times n}(\mathbb{R}) \to M_{n \times m}(\mathbb{R})$ defined by $T(A) = A^T$
- ▶ $T: P_1 \rightarrow P_2$ defined by $T(a+bx) = (a+2b) + 3ax + 4bx^2$



Definition

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Examples

- 1. Verify that $T:M_{m\times n}(\mathbb{R})\to M_{n\times m}(\mathbb{R}),$ where $T(A)=A^T,$ is a linear transformation.
 - The transpose of an $m \times n$ matrix is an $n \times m$ matrix.
 - ▶ If $A, B \in M_{m \times n}(\mathbb{R})$, then

 $T(A+B) = (A+B)^T = A^T + B^T = T(A) + T(B).$

- ▶ If $A \in M_{m \times n}(\mathbb{R})$ and $c \in \mathbb{R}$, then $T(cA) = (cA)^T = cA^T = cT(A).$
- 2. Verify that $T:C^k(I)\to C^{k-2}(I),$ where T(y)=y''+y, is a linear transformation.
 - If $y \in C^k(I)$ then $T(y) = y'' + y \in C^{k-2}(I)$.
 - ► If $y_1, y_2 \in C^k(I)$, then $T(y_1 + y_2) = (y_1 + y_2)'' + (y_1 + y_2) = y_1'' + y_2'' + y_1 + y_2$ $= (y_1'' + y_1) + (y_2'' + y_2) = T(y_1) + T(y_2).$
 - If $y \in C^k(I)$ and $c \in \mathbb{R}$, then T(cy) = (cy)'' + (cy) = cy'' + cy = c(y'' + y) = cT(y).

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Specifying linear transformations

A consequence of the properties of a linear transformation is that they preserve linear combinations, in the sense that

$$T(c_1\mathbf{v}_1 + \dots + c_n\mathbf{v}_n) = c_1 T(\mathbf{v}_1) + \dots + c_n T(\mathbf{v}_n).$$

In particular, if $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is a basis for the domain of T, then knowing $T(\mathbf{v}_1), \ldots, T(\mathbf{v}_n)$ is enough to determine T everywhere.



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Composition of linear trans. Kernel and Range Let A be an $m \times n$ matrix with real entries and define $T : \mathbb{R}^n \to \mathbb{R}^m$ by $T(\mathbf{x}) = A\mathbf{x}$. Verify that T is a linear transformation.

If x is an n × 1 column vector then Ax is an m × 1 column vector.

$$T(\mathbf{x} + \mathbf{y}) = A(\mathbf{x} + \mathbf{y}) = A\mathbf{x} + A\mathbf{y} = T(\mathbf{x}) + T(\mathbf{y})$$

$$\bullet \ T(c\mathbf{x}) = A(c\mathbf{x}) = cA\mathbf{x} = cT(\mathbf{x})$$

Such a transformation is called a **matrix transformation**. In fact, *every* linear transformation from \mathbb{R}^n to \mathbb{R}^m is a matrix transformation.



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Example

Determine the matrix of the linear transformation $T:\mathbb{R}^4\to\mathbb{R}^3$ defined by

$$T(x_1, x_2, x_3, x_4) = (2x_1 + 3x_2 + x_4, 5x_1 + 9x_3 - x_4, 4x_1 + 2x_2 - x_3 + 7x_4).$$

Theorem

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then T is described by the matrix transformation $T(\mathbf{x}) = A\mathbf{x}$, where

$$A = \begin{bmatrix} T(\mathbf{e}_1) & T(\mathbf{e}_2) & \cdots & T(\mathbf{e}_n) \end{bmatrix}$$



and $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n$ denote the standard basis vectors for \mathbb{R}^n . This A is called the matrix of T.

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Definition

Suppose $T: V \to W$ is a linear transformation. The set consisting of all the vectors $\mathbf{v} \in V$ such that $T(\mathbf{v}) = \mathbf{0}$ is called the **kernel** of T. It is denoted

$$\operatorname{Ker}(T) = \{ \mathbf{v} \in V : T(\mathbf{v}) = \mathbf{0} \}.$$

Example

Let $T: C^k(I) \to C^{k-2}(I)$ be the linear transformation T(y) = y'' + y. Its kernel is spanned by $\{\cos x, \sin x\}$.

Remarks

The kernel of a linear transformation is a subspace of its domain.



The kernel of a matrix transformation is simply the null space of the matrix.

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Definition

The **range** of the linear transformation $T: V \rightarrow W$ is the subset of W consisting of everything "hit by" T. In symbols,

$$\operatorname{Rng}(T) = \{T(\mathbf{v}) \in W : \mathbf{v} \in V\}.$$

Example

Consider the linear transformation $T: M_n(\mathbb{R}) \to M_n(\mathbb{R})$ defined by $T(A) = A + A^T$. The range of T is the subspace of symmetric $n \times n$ matrices.

Remarks

The range of a linear transformation is a subspace of its codomain.



 The range of a matrix transformation is the column space of the matrix.

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- ▶ $\operatorname{Ker}(T) = \operatorname{nullspace}(A)$,
- ▶ $\operatorname{Rng}(T) = \operatorname{colspace}(A)$,
- the domain of T is \mathbb{R}^n .

- $\dim(\operatorname{Ker}(T)) = \operatorname{nullity}(A)$,
- $\dim(\operatorname{Rng}(T)) = \operatorname{rank}(A)$,
- dim (domain of T) = n.

We know from the rank-nullity theorem that

 $\operatorname{rank}(A) + \operatorname{nullity}(A) = n.$

This fact is also true when T is not a matrix transformation:

Theorem

If $T:V \rightarrow W$ is a linear transformation and V is finite-dimensional, then

 $\dim\left(\operatorname{Ker}(T)\right) + \dim\left(\operatorname{Rng}(T)\right) = \dim(V).$



Rank-Nullity revisited

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The function of bases

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Theorem

Let V be a vector space with basis $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$. Then every vector $\mathbf{v} \in V$ can be written in a unique way as a linear combination

$$\mathbf{v} = c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 + \dots + c_n \mathbf{v}_n.$$

In other words, picking a basis for a vector space allows us to give coordinates for points. This will allow us to give matrices for linear transformations of vector spaces besides \mathbb{R}^n .



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The matrix of a linear transformation

Definition

Let V and W be vector spaces with *ordered* bases $B = {\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n}$ and $C = {\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_m}$, respectively, and let $T: V \to W$ be a linear transformation. The matrix representation of T relative to the bases B and C is

$$A = [a_{ij}]$$

where

$$T(\mathbf{v}_j) = a_{1j}\mathbf{w}_1 + a_{2j}\mathbf{w}_2 + \dots + a_{mj}\mathbf{w}_m.$$

In other words, A is the matrix whose j-th column is $T(\mathbf{v}_j)$, expressed in coordinates using $\{\mathbf{w}_1, \ldots, \mathbf{w}_m\}$.



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Let $T: P_1 \to P_2$ be the linear transformation defined by

$$T(a+bx) = (2a-3b) + (b-5a)x + (a+b)x^{2}.$$

Example

Use bases $\{1, x\}$ for P_1 and $\{1, x, x^2\}$ for P_2 to give a matrix representation of T. We have

$$T(1) = 2 - 5x + x^2$$
 and $T(x) = -3 + x + x^2$,

SO

$$A_1 = \begin{bmatrix} 2 & -3 \\ -5 & 1 \\ 1 & 1 \end{bmatrix}.$$



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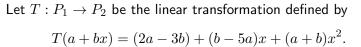
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Example

Now instead use the bases $\{1,x+5\}$ for P_1 and $\{1,1+x,1+x^2\}$ for $P_2.$ We have

$$T(1) = 2 - 5x + x^{2} = 6(1) - 5(1 + x) + (1 + x^{2})$$

and

$$T(x+5) = 7 - 24x + 6x^{2} = 25(1) - 24(1+x) + 6(1+x^{2}),$$

so

$$A_1 = \begin{bmatrix} 2 & -3 \\ -5 & 1 \\ 1 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 6 & 25 \\ -5 & -24 \\ 1 & 6 \end{bmatrix}.$$



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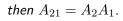
Composition of linear transformations

Definition

Let $T_1: U \to V$ and $T_2: V \to W$ be linear transformations. Their **composition** is the linear transformation $T_2 \circ T_1$ defined by $(T_2 \circ T_1) (\mathbf{u}) = T_2 (T_1(\mathbf{u})).$

Let T_1 and T_2 be as above, and let B, C, and D be ordered bases for U, V, and W, respectively. If

- A_1 is the matrix representation for T_1 relative to B and C,
- A_2 is the matrix representation for T_2 relative to C and D,
- ► A₂₁ is the matrix representation for T₂ ∘ T₁ relative to B and D,



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The inverse of a linear transformation

Definition

If $T: V \to W$ is a linear transformation, its **inverse** (if it exists) is a linear transformation $T^{-1}: W \to V$ such that

 $\left(T^{-1}\circ T\right)\left(\mathbf{v}
ight)=\mathbf{v}$ and $\left(T\circ T^{-1}
ight)\left(\mathbf{w}
ight)=\mathbf{w}$

for all $\mathbf{v} \in V$ and $\mathbf{w} \in W$.

Theorem

Let T be as above and let A be the matrix representation of T relative to bases B and C for V and W, respectively. T has an inverse transformation if and only if A is invertible and, if so, T^{-1} is the linear transformation with matrix A^{-1} relative to C and B.



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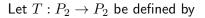
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$$T(a + bx + cx^{2}) = (3a - b + c) + (a - c)x + (4b + c)x^{2}.$$

Example

Using the basis $\{1,x,x^2\}$ for P_2 , the matrix representation for T is

$$A = \begin{bmatrix} 3 & -1 & 1 \\ 1 & 0 & -1 \\ 0 & 4 & 1 \end{bmatrix}$$

.

This matrix is invertible and

$$A^{-1} = \frac{1}{17} \begin{bmatrix} 4 & 5 & 1 \\ -1 & 3 & 4 \\ 4 & -12 & 1 \end{bmatrix}$$

Thus, T^{-1} is given by

$$T^{-1}(a+bx+cx^2) = \frac{4a+5b+c}{17} + \frac{-a+3b+4c}{17}x + \frac{4a-12b+c}{17}x^2.$$



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Theorem

Let $T: V \to W$ be a linear transformation and A be a matrix representation of T relative to some bases for V and W.

- ► Ker $(T) = \{c_1 \mathbf{v}_1 + \dots + c_n \mathbf{v}_n \in V : (c_1, \dots, c_n) \in$ nullspace $(A)\},$
- ► Rng(T) = { c_1 **w**₁ + · · · + c_m **w**_m ∈ W : (c_1 , . . . , c_m) ∈ colspace(A)}.

