Math 240

The Utility o Bases

The Rank-Nullity Theorem

Homogeneous linear systems

Nonhomogeneous linear systems

On Bases and the Rank-Nullity Theorem

Math 240 — Calculus III

Summer 2015, Session II

Tuesday, July 14, 2015



Agenda

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2. The Rank-Nullity Theorem Homogeneous linear systems Nonhomogeneous linear systems



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Let V be a vector space with basis $\mathbf{v}_1, \ldots, \mathbf{v}_n$.

 \blacktriangleright Since the basis is a spanning set, every $\mathbf{v} \in V$ can be written as

$$\mathbf{v} = c_1 \mathbf{v}_1 + \dots + c_n \mathbf{v}_n,$$

What are bases good for?

Since the basis is independent, if

$$\mathbf{v} = d_1 \mathbf{v}_1 + \dots + d_n \mathbf{v}_n$$

is another way of writing **v**, then $c_i = d_i$ for each *i*.

Theorem

Every vector $\mathbf{v} \in V$ can be written in a unique way as a linear combination of basis elements.

The theorem allows us to identify \mathbf{v} with the vector (c_1, \ldots, c_n) .



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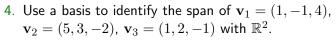
1. What is a basis for the vector space whose vectors are complex numbers and whose scalars are real numbers? Is there more than one "natural" one?

Examples

- 2. In the subspace of $C^0(\mathbb{R})$ spanned by $f_1(x) = 2\sin^2 x$ and $f_2(x) = -5\cos^2 x$, give coordinates for the vector f(x) = 1.
- 3. In the vector space P_2 , show that

$$p_1(x) = 1 - x$$
, $p_2(x) = 5 + 3x - 2x^2$, $p_3(x) = 1 + 2x - x^2$

is a basis. Then write the standard basis vectors $e_1(x) = 1$, $e_2(x) = x$, $e_3(x) = x^2$ in terms of $p_1(x)$, $p_2(x)$, and $p_3(x)$.





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Homogeneous linear systems Nonhomogeneous linear systems Suppose that we have vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k \in \mathbb{R}^n$. Form the matrix $A = [\mathbf{v}_1 \cdots \mathbf{v}_k]$. We saw yesterday that

▶ if rank A < k then our vectors are linearly dependent,</p>

Reinterpreting rank

• if $\operatorname{rank} A = k$ then the vectors are linearly independent.

Specifically, we get a linear dependency for each independent vector in $\operatorname{nullspace}(A)$. The remaining vectors will be a basis for their span.

Proposition

The rank of a matrix is equal to the dimension of the span of its columns.

Definition

The span of the columns of a matrix A is called its **column** space. It is denoted by colspace(A).



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Definition

When A is an $m\times n$ matrix, recall that the null space of A is

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nullspace
$$(A) = \{ \mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = \mathbf{0} \}.$$

Its dimension is referred to as the **nullity** of A.

Theorem (Rank-Nullity Theorem) For any $m \times n$ matrix A,

 $\operatorname{rank}(A) + \operatorname{nullity}(A) = n.$



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Homogeneous linear systems

We're now going to examine the geometry of the solution set of a linear system. Consider the linear system

 $A\mathbf{x} = \mathbf{b},$

where A is $m \times n$.

If b = 0, the system is called **homogeneous**. In this case, the solution set is simply the null space of A.

Any homogeneous system has the solution $\mathbf{x} = \mathbf{0}$, which is called the **trivial solution**. Geometrically, this means that the solution set passes through the origin. Furthermore, we have shown that the solution set of a homogeneous system is in fact a subspace of \mathbb{R}^n .



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Structure of a homogeneous solution set

Theorem

- If rank(A) = n, then Ax = 0 has only the trivial solution x = 0, so nullspace(A) = {0}.
- If rank(A) = r < n, then Ax = 0 has an infinite number of solutions, all of which are of the form

$$\mathbf{x} = c_1 \mathbf{x}_1 + c_2 \mathbf{x}_2 + \dots + c_{n-r} \mathbf{x}_{n-r},$$

where $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_{n-r}\}$ is a basis for $\operatorname{nullspace}(A)$.

Remark

Such an expression is called the **general solution** to the homogeneous linear system.



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Nonhomogeneous linear systems Now consider a nonhomogeneous linear system

 $A\mathbf{x} = \mathbf{b}$

where A be an $m \times n$ matrix and b is not necessarily 0.

Theorem

- ▶ If **b** is not in colspace(A), then the system is inconsistent.
- If $\mathbf{b} \in \operatorname{colspace}(A)$, then the system is consistent and has
 - a unique solution if and only if rank(A) = n.
 - ▶ an infinite number of solutions if and only if rank(A) < n.

Geometrically, a nonhomogeneous solution set is just the corresponding homogeneous solution set that has been shifted away from the origin.



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Structure of a nonhomogeneous solution set

Theorem

In the case where rank(A) = r < n and $\mathbf{b} \in colspace(A)$, then all solutions are of the form

$$\mathbf{x} = \underbrace{c_1 \mathbf{x}_1 + c_2 \mathbf{x}_2 + \dots + c_{n-r} \mathbf{x}_{n-r}}_{\mathbf{x}_c} + \mathbf{x}_p,$$

where \mathbf{x}_p is any particular solution to $A\mathbf{x} = \mathbf{b}$ and $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_{n-r}\}$ is a basis for $\operatorname{nullspace}(A)$.

Remark

The above expression is the **general solution** to a nonhomogeneous linear system. It has two components:

- the complementary solution, \mathbf{x}_c , and
- the particular solution, \mathbf{x}_p .

