The Determinant

Math 240 — Calculus III

Summer 2015, Session II

Wednesday, July 8, 2015

Agenda

1. Definition of the determinant
2. Computing determinants
3. Properties of determinants

What is the determinant?

Yesterday: $A\mathbf{x} = \mathbf{b}$ has a unique solution when A is square and nonsingular.
Today: how to determine whether A is invertible.

Example

Recall that a 2×2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is invertible as long as $ad - bc \neq 0$. The quantity $ad - bc$ is the **determinant** of this matrix and the matrix is invertible exactly when its determinant is nonzero.

What should the determinant be?

- We want to associate a number with a matrix that is zero if and only if the matrix is singular.
- An $n \times n$ matrix is nonsingular if and only if its rank is n.
- For upper triangular matrices, the rank is the number of nonzero entries on the diagonal.
- To determine if every number in a set is nonzero, we can multiply them.

Definition

The **determinant** of an upper triangular matrix, $A = [a_{ij}]$, is the product of the elements a_{ii} along its main diagonal. We write

$$\det(A) = \begin{vmatrix} a_{11} & \cdots & a_{1n} \\ & \ddots & \vdots \\ 0 & & a_{nn} \end{vmatrix} = a_{11}a_{22}\cdots a_{nn}.$$

What should the determinant be?

What about matrices that are not upper triangular? We can make any matrix upper triangular via row reduction. So how do elementary row operations affect the determinant?

- $M_i(k)$ multiplies the determinant by k. (Remember that k cannot be zero.)
- $A_{ij}(k)$ does not change the determinant.
- P_{ij} multiplies the determinant by -1.

Let's extend these properties to *all matrices*.

Definition

The **determinant** of a square matrix, A, is the determinant of any upper triangular matrix obtained from A by row reduction times $\frac{1}{k}$ for every $M_i(k)$ operation used while reducing as well as -1 for each P_{ij} operation used.

Computing determinants

Example

Compute $\det(A)$, where $A = \begin{bmatrix} 0 & 2 & 1 \\ 2 & 3 & 10 \\ 1 & -1 & 0 \end{bmatrix}$.

We need to put A in upper triangular form.

$$\begin{bmatrix} 0 & 2 & 1 \\ 2 & 3 & 10 \\ 1 & -1 & 0 \end{bmatrix} \xrightarrow[\substack{M_2\left(\frac{1}{5}\right) \\ A_{23}(-2)}]{\substack{P_{13} \\ A_{12}(-2)}} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & -3 \end{bmatrix}$$

So the determinant is

$$\begin{vmatrix} 0 & 2 & 1 \\ 2 & 3 & 10 \\ 1 & -1 & 0 \end{vmatrix} = (-1)(5) \begin{vmatrix} 1 & -1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & -3 \end{vmatrix} = 15.$$

Computing determinants

Important Example

Given a general 2×2 matrix, $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, compute $\det(A)$.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \xrightarrow{A_{12}\left(-\frac{c}{a}\right)} \begin{bmatrix} a & b \\ 0 & d - \frac{bc}{a} \end{bmatrix}$$

so

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ 0 & d - \frac{bc}{a} \end{vmatrix} = ad - bc.$$

This explains

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$ when $ad - bc \neq 0$.

Other methods of computing determinants

Theorem (Cofactor expansion)

Suppose $A = [a_{ij}]$ is an $n \times n$ matrix. For any fixed k between 1 and n,

$$\det(A) = \sum_{j=1}^{n} (-1)^{k+j} a_{kj} \det(A_{kj}) = \sum_{i=1}^{n} (-1)^{i+k} a_{ik} \det(A_{ik})$$

where A_{ij} is the $(n-1) \times (n-1)$ submatrix obtained by removing the i^{th} row and j^{th} column from A.

Example

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a & b & c \\ d & e & f \end{vmatrix} = \begin{vmatrix} b & c \\ e & f \end{vmatrix} \mathbf{i} - \begin{vmatrix} a & c \\ d & f \end{vmatrix} \mathbf{j} + \begin{vmatrix} a & b \\ d & e \end{vmatrix} \mathbf{k}.$$

Other methods of computing determinants

Corollary

If $A = [a_{ij}]$ is an $n \times n$ matrix and the element a_{ij} is the only nonzero entry in its row or column then

$$\det(A) = (-1)^{i+j} a_{ij} \det(A_{ij}).$$

Example

$$\begin{vmatrix} 0 & 2 & 1 \\ 3 & 0 & 0 \\ 0 & 1 & 5 \end{vmatrix} = -3 \begin{vmatrix} 2 & 1 \\ 1 & 5 \end{vmatrix} = -27.$$

Other methods of computing determinants

Some of you may have learned the method of computing a 3×3 determinant by multiplying diagonals.

$$\begin{array}{cccccc} & & & - & - & - \\ a_{11} & a_{12} & a_{13} & a_{11} & a_{12} & \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} & \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} & \\ & & & + & + & + \end{array}$$

Be aware that this method does not work for matrices larger than 3×3.

Properties of determinants

Theorem (Main theorem)

Suppose A is an $n \times n$ matrix. The following are equivalent:

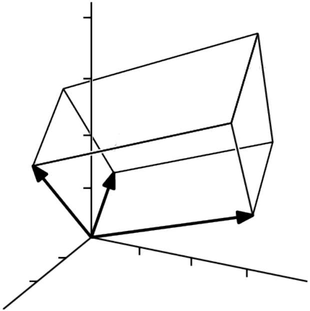
- *A is invertible,*
- *$\det(A) \neq 0$.*

Further properties

- $\det\left(A^T\right) = \det(A)$.
- The determinant of a *lower* triangular matrix is also the product of the elements on the main diagonal.
- If A has a row or column of zeros then $\det(A) = 0$.
- If two rows or columns of A are the same then $\det(A) = 0$.
- $\det(AB) = \det(A)\det(B)$, $\det\left(A^{-1}\right) = \det(A)^{-1}$.
- $\det(sA) = s^n \det(A)$.
- It is not true that $\det(A+B) = \det(A) + \det(B)$.

Geometric interpretation

Let A be an $n \times n$ matrix and $\mathbf{a}_1, \ldots, \mathbf{a}_n$ be the rows or columns of A.



Source: `en.wikibooks.org/wiki/Linear_Algebra`

Theorem

The volume (or area, if $n = 2$) of the paralellepiped determined by the vectors $\mathbf{a}_1, \ldots, \mathbf{a}_n$ is $|\det(A)|$.

Corollary

The vectors $\mathbf{a}_1, \ldots, \mathbf{a}_n$ lie in the same hyperplane if and only if $\det(A) = 0$.