Final Exam Practice Problems — Answers

Math 240 — Calculus III

Summer 2015, Session II

Linear Algebra

1. (a) $\{(0,0) \in \mathbb{R}^2\}$
   (b) $\{(-t, 4t, t) \in \mathbb{R}^3 : t \in \mathbb{R}\}$
   (c) $\{(1,2,4) \in \mathbb{R}^3\}$
   (d) This system is inconsistent.
   (e) $\{(-1-2s-4t, s, -2-3t, t) \in \mathbb{R}^4 : s,t \in \mathbb{R}\}$ (Answers may vary.)

2. $\mathbf{x}(t) = (3-2t,\ 1-t,\ t)$

3. (a) -7
   (b) 29
   (c) -2

4. (a) The inverse of this matrix is
$$\frac{1}{2}\begin{bmatrix} 2 & -4 \\ -1 & 3 \end{bmatrix}.$$
   (b) This matrix is not invertible.
   (c) The inverse of this matrix is
$$\begin{bmatrix} -3 & 0 & -4 \\ -2 & 1 & -3 \\ 1 & 0 & 1 \end{bmatrix}.$$

5. (a) Answers may vary. Correct answers include
$$\{\mathbf{v}_1,\ \mathbf{v}_3\} \quad \text{and} \quad \{(1,2,1,3),\ (0,1,4,1)\}.$$
   (b) Answers may vary. Correct answers include
$$\{\mathbf{v}_1,\ \mathbf{v}_2,\ \mathbf{v}_4\} \quad \text{and} \quad \{(1,1,1,1,1),\ (0,0,1,3,0),\ (0,0,0,1,0)\}.$$

6. One method of verification is to compute
$$\det\left(\begin{bmatrix}\mathbf{v}_1 & \mathbf{v}_2\end{bmatrix}\right) = \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = 1$$
and note that it is nonzero. Then
$$(2,-1) = 5\mathbf{v}_1 - 3\mathbf{v}_2.$$

7. A basis for $\operatorname{Ker}(T)$ is $\{(-2,1,1)\}$. A basis for $\operatorname{Rng}(T)$ is $\{(1,-2,3),\ (0,1,-2)\}$. (Answers may vary.)

8. (a) $\begin{bmatrix} 1 & -2 \\ 3 & 0 \end{bmatrix}$

   (b) $\begin{bmatrix} -12 & -18 \\ 9 & 13 \end{bmatrix}$

9. $\begin{bmatrix} 5 & 1 \\ -1 & 5 \end{bmatrix}$

10. (a) False.
    (b) True.
    (c) True.
    (d) True.
    (e) False.

More Linear Algebra

11. $[D] = \begin{bmatrix} -3 & 2 & 1 & 0 \\ -2 & -3 & 0 & 1 \\ 0 & 0 & -3 & 2 \\ 0 & 0 & -2 & -3 \end{bmatrix}, \quad [L] = \begin{bmatrix} 4 & 60 & 22 & -40 \\ -60 & 4 & 40 & 22 \\ 0 & 0 & 4 & 60 \\ 0 & 0 & -60 & 4 \end{bmatrix}$

12. $x - 2y + z = 0$

13. (a) $3\begin{bmatrix} 1 & 0 & 3 \\ 0 & 2 & -1 \end{bmatrix} + \begin{bmatrix} 0 & 5 & -1 \\ 2 & 2 & 0 \end{bmatrix} - \begin{bmatrix} 3 & 5 & 8 \\ 2 & 8 & -3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

    (b) $(x^3+7x+3) - (x^3+7x^2+3) - (x^3-x^2-x) + (x^3+6x^2-8x) = 0$

14. (a) This is a basis.
    (b) Not a basis.
    (c) Not a basis — $C^0(\mathbb{R})$ is infinite dimensional.

15. $\begin{bmatrix} 1 & 0 & 9 \\ 0 & -5 & 0 \\ -1 & 0 & 0 \end{bmatrix}$

16. We can see from the definition that T takes a 2×2 matrix as input and produces a 2×3 matrix. Now we check that it preserves addition and scalar multiplication:

$$\begin{aligned} T\left(\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} + \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}\right) &= T\left(\begin{bmatrix} a_1+a_2 & b_1+b_2 \\ c_1+c_2 & d_1+d_2 \end{bmatrix}\right) \\ &= \begin{bmatrix} a_1+a_2 & a_1+a_2 & a_1+a_2-b_1-b_2 \\ c_1+c_2 & c_1+c_2 & c_1+c_2-d_1-d_2 \end{bmatrix} \end{aligned}$$

and

$$T\left(\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}\right) + T\left(\begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}\right) = \begin{bmatrix} a_1 & a_1 & a_1 - b_1 \\ c_1 & c_1 & c_1 - d_1 \end{bmatrix} + \begin{bmatrix} a_2 & a_2 & a_2 - b_2 \\ c_2 & c_2 & c_2 - d_2 \end{bmatrix}$$
$$= \begin{bmatrix} a_1 + a_2 & a_1 + a_2 & a_1 - b_1 + a_2 - b_2 \\ c_1 + c_2 & c_1 + c_2 & c_1 - d_1 + c_2 - d_2 \end{bmatrix}.$$

Since these results are the same, addition is preserved. For scalar multiplication,

$$T\left(s\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = T\left(\begin{bmatrix} sa & sb \\ sc & sd \end{bmatrix}\right) = \begin{bmatrix} sa & sa & sa - sb \\ sc & sc & sc - sd \end{bmatrix}$$

and

$$sT\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = s\begin{bmatrix} a & a & a - b \\ c & c & c - d \end{bmatrix} = \begin{bmatrix} sa & sa & s(a-b) \\ sc & sc & s(c-d) \end{bmatrix}.$$

Since these results are the same, scalar multiplication is preserved. Thus, T is a linear transformation. This transformation has a trivial kernel because if

$$\begin{bmatrix} a & a & a - b \\ c & c & c - d \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

then $a = 0$ and $c = 0$, hence also $-b = 0$ and $-d = 0$. One basis for its range is

$$\left\{ \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}.$$

17. (a) $\lambda = 4, 4; \mathbf{v} = (3, 2)$

    (b) $\lambda_1 = 2, 2; \mathbf{v}_1 = (-1, 1, 3); \lambda_2 = -3; \mathbf{v}_2 = (-1, 1, -2)$

18. $\begin{bmatrix} -1 & -2 & -2 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix}$

19. $k = 1$ or $k = 5$

20. The matrix is not diagonalizable; its Jordan form is

$$\begin{bmatrix} 5 & 0 & 0 \\ 0 & 4 & 1 \\ 0 & 0 & 4 \end{bmatrix}.$$

Differential Equations

1. Convert the given equation into a first-order system of equations.

   (a) $\mathbf{x}' = \begin{bmatrix} 0 & 1 \\ -4 & -4 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ \cos t \end{bmatrix}$, where $\mathbf{x} = \begin{bmatrix} y \\ y' \end{bmatrix}$

   (b) $\mathbf{x}' = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix}\mathbf{x}$, where $\mathbf{x} = \begin{bmatrix} y \\ y' \\ y'' \end{bmatrix}$

2. $\mathbf{x}(t) = \frac{1}{2}\begin{bmatrix} e^{3t} + e^{-t} \\ 5e^{3t} + e^{-t} \end{bmatrix}$

3. $\mathbf{x}(t) = c_1 e^t \begin{bmatrix} \cos 2t \\ \cos 2t + \sin 2t \end{bmatrix} + c_2 e^t \begin{bmatrix} \sin 2t \\ \sin 2t - \cos 2t \end{bmatrix}$

4. $\mathbf{x}(t) = \frac{1}{5}e^{-t}\begin{bmatrix} 5\cos t \\ 2\cos t + \sin t \end{bmatrix} - \frac{3}{5}e^{-t}\begin{bmatrix} 5\sin t \\ 2\sin t - \cos t \end{bmatrix} = e^{-t}\begin{bmatrix} \cos t - 3\sin t \\ \cos t - \sin t \end{bmatrix}$

5. $\mathbf{x}(t) = c_1 e^t \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 e^t \begin{bmatrix} 2t+1 \\ t \end{bmatrix} = e^t \begin{bmatrix} 2c_1 + c_2(2t+1) \\ c_1 + c_2 t \end{bmatrix}$

6. $\mathbf{x}(t) = c_1 \begin{bmatrix} 3 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 3t+1 \\ -t \end{bmatrix} = \begin{bmatrix} 3c_1 + c_2(3t+1) \\ -c_1 - c_2 t \end{bmatrix}$

7. $\mathbf{x}(t) = c_1 e^{-2t}\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} + c_2 e^{-t}\begin{bmatrix} \sqrt{2}\cos\sqrt{2}t \\ \sin\sqrt{2}t \\ \sqrt{2}\cos\sqrt{2}t - \sin\sqrt{2}t \end{bmatrix} + c_3 e^{-t}\begin{bmatrix} \sqrt{2}\sin\sqrt{2}t \\ -\cos\sqrt{2}t \\ \sqrt{2}\sin\sqrt{2}t + \cos\sqrt{2}t \end{bmatrix}$

8. $\mathbf{x}(t) = c_1 e^{-2t}\begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} + c_2 e^{-2t}\begin{bmatrix} 4 \\ -2 \\ -4 \end{bmatrix} + c_3 e^{-2t}\begin{bmatrix} 4t+1 \\ -2t \\ -4t \end{bmatrix}$

9. (a) $y(x) = c_1 x^2 + c_2 x^{-4}$
   (b) $y(x) = c_1 + c_2 x^{\sqrt{7}} + c_3 x^{-\sqrt{7}}$

10. (a) $A(D) = D^2(D-1)$
    (b) $A(D) = (D-7)^4(D^2+16)$
    (c) $A(D) = (D-4)^2(D^2-8D+41)D^2(D^2+4D+5)^3$
    (d) $A(D) = D^2 + 6D + 10$

11. (a) $y(x) = c_1 e^{-2x} + c_2 x e^{-2x}$
    (b) $y(x) = c_1 e^{-4x}\cos 2x + c_2 e^{-4x}\sin 2x$
    (c) $y(x) = c_1 e^{-2x} + c_2 e^{2x} + c_3 x e^{2x}$
    (d) $y(x) = c_1 e^{-x} + c_2 e^{2x} + \frac{5}{3}e^{2x}$
    (e) $y(t) = c_1 \cos 2t + c_2 \sin 2t + \left(\frac{13}{32}t - \frac{1}{12}t^3\right)\cos 2t + \left(\frac{7}{4}t + \frac{13}{16}t^2\right)\sin 2t$
    (f) $y(t) = c_1 e^{-2t} + c_2 t e^{-2t} + \frac{1}{6}t^3 e^{-2t} - \frac{1}{8}e^{2t}$

12. (a) $y(t) = (2-t)e^{4t}$
    (b) $y(t) = e^t - \cos t$

13. (a) $y(t) = e^{-t}(\cos 2t + 2\sin 2t) = \sqrt{5}e^{-t}\cos(2t - \arctan 2)$. This system is underdamped.
    (b) $y(t) = -\frac{1}{4}e^{-t/2} + \frac{5}{4}e^{-5t/2} = e^{-3t/2}\left(-\frac{1}{4}e^{t} + \frac{5}{4}e^{-t}\right)$. This system is overdamped.
    (c) $y(t) = e^{-2t} - 2e^{-3t} = e^{-5t/2}\left(e^{t/2} - 2e^{-t/2}\right)$. This system is overdamped.

14. $y(t) = 16\cos\frac{3}{4}t + 12\sin\frac{3}{4}t - 16\cos 2t$

15. (a) $y(x) = (c_1 + c_2 t)e^{3t} + 4t^{5/2}e^{3t}$
    (b) $y(x) = c_1 e^x + c_2 x e^x - e^x \ln x$