Final Exam Practice Problems

Math 240 — Calculus III

Summer 2015, Session II

Linear Algebra

1. Find the solution set of the given linear system.

   (a) $$\begin{aligned} -5x_1 - 4x_2 &= 0 \\ -8x_1 + x_2 &= 0 \end{aligned}$$

   (b) $$\begin{aligned} -x_1 + 6x_2 - 25x_3 &= 0 \\ 9x_1 + 6x_2 - 15x_3 &= 0 \end{aligned}$$

   (c) $$\begin{aligned} x_1 - 2x_2 + 2x_3 &= 5 \\ x_1 - x_2 &= -1 \\ -x_1 + x_2 + x_3 &= 5 \end{aligned}$$

   (d) $$\begin{aligned} x_1 + x_2 + 3x_3 &= 3 \\ -x_1 + x_2 + x_3 &= -1 \\ 2x_1 + 3x_2 + 8x_3 &= 4 \end{aligned}$$

   (e) $$\begin{aligned} x_1 + 2x_2 - x_3 + x_4 &= 1 \\ -x_1 - 2x_2 + 3x_3 + 5x_4 &= -5 \\ -x_1 - 2x_2 - x_3 - 7x_4 &= 3 \end{aligned}$$

2. Parameterize the line that is the intersection of the planes $x+y+3z=4$ and $x+2y+4z=5$.

3. Calculate the determinant of the given matrix.

   (a) $\begin{bmatrix} 2 & 3 \\ 3 & 1 \end{bmatrix}$

   (b) $\begin{bmatrix} -1 & 2 & 0 \\ 2 & -2 & 5 \\ 4 & -1 & 3 \end{bmatrix}$

   (c) $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 5 & 8 \\ 1 & 1 & 2 & 3 \\ 1 & 3 & 5 & 8 \end{bmatrix}$

4. Determine whether the given matrix is invertible. Find its inverse if it has one.

   (a) $\begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}$

   (b) $\begin{bmatrix} 1 & 3 & 1 \\ 2 & 5 & 2 \\ 4 & 7 & 4 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 0 & 4 \\ -1 & 1 & -1 \\ -1 & 0 & -3 \end{bmatrix}$

5. Find a basis for the subset of $\mathbb{R}^n$ spanned by the given vectors.

   (a) $\mathbf{v}_1 = (1,2,1,3),\ \mathbf{v}_2 = (3,6,3,9),\ \mathbf{v}_3 = (1,3,5,4),\ \mathbf{v}_4 = (2,3,-2,5)$

   (b) $\mathbf{v}_1 = (1,1,1,1,1)$, $\mathbf{v}_2 = (1,1,2,4,1)$, $\mathbf{v}_3 = (0,0,1,3,0)$, $\mathbf{v}_4 = (0,0,1,4,0)$

6. Let $\mathbf{v}_1 = (1,1)$ and $\mathbf{v}_2 = (1,2)$. Verify that $\{\mathbf{v}_1, \mathbf{v}_2\}$ is a basis for $\mathbb{R}^2$ and express $(2,-1)$ in this basis.

7. Determine a basis for the kernel and range of the linear transformation $T(\mathbf{v}) = A\mathbf{v}$ where

$$A = \begin{bmatrix} 1 & 0 & 2 \\ -2 & 1 & -5 \\ 3 & -2 & 8 \end{bmatrix}.$$

8. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation $T(a,b) = (a-2b, 3a)$. Find the matrix representation of T relative to the given ordered basis.

   (a) the standard basis $\{(1,0),\ (0,1)\}$

   (b) $\{(1,2),\ (1,3)\}$

9. Let V be the subspace of $C^\infty(\mathbb{R})$ spanned by $y_1 = e^{2x}\cos x$ and $y_2 = e^{2x}\sin x$. Find the matrix representation of the linear transformation $T : V \to V$ given by $T(f) = f' + 3f$ relative to the ordered basis $\{y_1, y_2\}$.

10. Determine whether the statement is true or false.

    (a) The set of invertible $n \times n$ matrices is a subspace of $M_n(\mathbb{R})$.

    (b) The set $\{(a,b,0,a) \in \mathbb{R}^4 : a,b \in \mathbb{R}\}$ is a subspace of $\mathbb{R}^4$.

    (c) The mapping $T : C^2(\mathbb{R}) \to C^0(\mathbb{R})$ defined by $T(f) = f'' - 3f' + 5f$ is a linear transformation.

    (d) If the standard basis vectors $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ are eigenvectors of an $n \times n$ matrix, then the matrix is diagonal.

    (e) If 1 is the only eigenvalue of an $n \times n$ matrix, then it must be the identity matrix.

More Linear Algebra

11. Let $V = \operatorname{span}\{e^{-3x}\cos 2x, e^{-3x}\sin 2x, xe^{-3x}\cos 2x, xe^{-3x}\sin 2x\}$, a subspace of $C^3(\mathbb{R})$. Let L be the differential operator $L = (D^2 - 2D + 3)(D+1)$. Write a matrix representation for the linear transformation $D : V \to V$ and use it to find a matrix for L.

12. Determine the equation for the plane in $\mathbb{R}^3$ spanned by

$$\{(1,2,3),\ (3,4,5),\ (4,5,6)\}.$$

13. (a) Find a linear dependence relation among the vectors

$$\begin{bmatrix} 1 & 0 & 3 \\ 0 & 2 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 5 & -1 \\ 2 & 2 & 0 \end{bmatrix}, \begin{bmatrix} -2 & 1 & 0 \\ 4 & 0 & 8 \end{bmatrix}, \begin{bmatrix} 3 & 5 & 8 \\ 2 & 8 & -3 \end{bmatrix}$$

in the vector space $M_{2\times 3}(\mathbb{R})$.

(b) Find a linear dependence relation among the vectors

$$x^3 + 7x + 3,\ x^3 + 7x^2 + 3,\ x^3 - x^2 - x,\ x^3 + 6x^2 - 8x$$

in the vector space P_3.

14. Determine whether the given set of vectors is a basis for the indicated vector space.

(a) $\mathbb{R}^3$, $\{(8, 4, 3),\ (1, -2, 1),\ (-1, 3, -1)\}$

(b) P_3, $\{x + 1,\ x^2 + x,\ x^3 + x^2,\ x^3 + 1\}$

(c) $C^0(\mathbb{R})$, $\{\cos x,\ \sin x,\ x\cos x,\ x\sin x\}$

15. Let T be the matrix transformation $T(\mathbf{x}) = A\mathbf{x}$ with

$$A = \begin{bmatrix} -2 & 15 & 0 \\ -1 & 3 & 26 \\ 0 & 0 & -5 \end{bmatrix}.$$

Use a change of basis matrix to write a matrix representation for T with respect to the basis $\{(5, 1, 0),\ (10, -2, 1),\ (0, 3, 0)\}$.

16. Consider the mapping $T : M_{2\times 2}(\mathbb{R}) \to M_{2\times 3}(\mathbb{R})$ defined by

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a & a & a-b \\ c & c & c-d \end{bmatrix}.$$

Show that T is a linear transformation and find bases for the kernel and range of T.

17. Find all eigenvalues and eigenvectors of the given matrix.

(a) $\begin{bmatrix} 10 & -9 \\ 4 & -2 \end{bmatrix}$

(b) $\begin{bmatrix} 3 & 4 & -1 \\ -1 & -2 & 1 \\ 3 & 9 & 0 \end{bmatrix}$

18. Use diagonalization to compute A^{25} where

$$A = \begin{bmatrix} -1 & -2 & -2 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix}.$$

19. Find all values of the constant k so that the matrix

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & k & 2 \\ 0 & 2 & k \end{bmatrix}$$

has at least one repeated eigenvalue.

20. Diagonalize the matrix if possible, otherwise find its Jordan Canonical Form.

$$A = \begin{bmatrix} 4 & 0 & 4 \\ 1 & 4 & 8 \\ 0 & 0 & 5 \end{bmatrix}$$

Differential Equations

1. Convert the given equation into a first-order system of equations.
   (a) $y'' + 4y' + 4y = \cos 2t$
   (b) $y''' - y' + y = 0$

2. Solve the initial value problem $\mathbf{x}' = A\mathbf{x}$, $\mathbf{x}(0) = (1, 3)$, where

$$A = \begin{bmatrix} -2 & 1 \\ -5 & 4 \end{bmatrix}.$$

3. Find the general solution to the vector differential equation

$$\mathbf{x}' = \begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix} \mathbf{x}.$$

4. Solve the initial value problem $\mathbf{x}' = A\mathbf{x}$, $\mathbf{x}(0) = (1, 1)$ where

$$A = \begin{bmatrix} 1 & -5 \\ 1 & -3 \end{bmatrix}.$$

5. Find the general solution to

$$\mathbf{x}' = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix} \mathbf{x}.$$

6. Solve the initial value problem $\mathbf{x}' = A\mathbf{x}$, $\mathbf{x}(0) = (2, 4)$ where

$$A = \begin{bmatrix} 3 & 9 \\ -1 & -3 \end{bmatrix}.$$

7. Solve

$$\mathbf{x}' = \begin{bmatrix} -3 & 0 & 2 \\ 1 & -1 & 0 \\ -2 & -1 & 0 \end{bmatrix} \mathbf{x}.$$

8. Solve

$$\mathbf{x}' = \begin{bmatrix} 2 & 4 & 2 \\ -2 & -4 & -1 \\ -4 & -4 & -4 \end{bmatrix} \mathbf{x}.$$

9. Determine the linearly independent solutions to the given differential equation of the form $y(x) = x^r$, and thereby determine the general solution to the differential equation on $(0, \infty)$.

(a) $x^2y'' + 3xy' - 8y = 0$, $x > 0$.

(b) $x^3y''' + 3x^2y'' - 6xy' = 0$, $x > 0$.

10. Determine an annihilator for the given function.

(a) $F(x) = 2e^x - 3x$

(b) $F(x) = x^3e^{7x} + 5\sin 4x$

(c) $F(x) = e^{4x}(x - 2\sin 5x) + 3x - x^2e^{-2x}\cos x$

(d) $F(x) = e^{-3x}(2\sin x + 7\cos x)$

11. Find the general solution to the differential equation.

(a) $(D + 2)^2y = 0$

(b) $y'' + 8y' + 20y = 0$

(c) $y''' - 2y'' - 4y' + 8y = 0$

(d) $y'' - y' - 2y = 5e^{2x}$

(e) $y'' + 4y = t^2\sin 2t + (6t + 7)\cos 2t$

(f) $y'' + 4y' + 4y = te^{-2t} - 2e^{2t}$

12. Solve the initial value problem.

(a) $y'' - 8y' + 16y = 0$, $y(0) = 2$, $y'(0) = 7$.

(b) $y''' - y'' + y' - y = 0$, $y(0) = 0$, $y'(0) = 1$, $y''(0) = 2$.

13. Determine the motion of the spring-mass system governed by the given initial-value problem. Determine if the motion is underdamped, critically damped, or overdamped and make a sketch of it. Include any intercepts with the coordinate axes.

(a) $y'' + 2y' + 5y = 0$, $y(0) = 1$, $y'(0) = 3$.

(b) $4y'' + 12y' + 5y = 0$, $y(0) = 1$, $y'(0) = -3$.

(c) $y'' + 5y' + 6y = 0$, $y(0) = -1$, $y'(0) = 4$.

14. Find an equation for and determine the period of the motion for the spring-mass system governed by the initial value problem

$$y'' + \frac{9}{16}y = 55\cos 2t, y(0) = 0, y'(0) = 9.$$

15. Find the general solution to the differential equation.

(a) $y'' - 6y' + 9y = 15e^{3x}\sqrt{x}$, $x > 0$.

(b) $y'' - 2y' + y = \frac{e^x}{x^2}$, $x > 0$.