Complex-Valued Trial Solutions

Math 240 — Calculus III

Summer 2013, Session II

Tuesday, August 6, 2013

Motivating example

Example

Find the general solution to

$$y'' + y' - 6y = 4\cos 2x.$$

1. Recall from yesterday that the complementary function is $y_c(x) = c_1 e^{-3x} + c_2 e^{2x}$.
2. The right-hand side would be annihilated by $D^2 + 4$.
3. Since $\pm 2i$ is not already a root of the auxiliary polynomial, use the trial solution $y_p(x) = c_3 \cos 2x + c_4 \sin 2x$.
4. Plugging y_p into the original equation yields $c_3 = -\frac{5}{13}$ and $c_4 = \frac{1}{13}$.
5. The general solution is

$$y(x) = c_1 e^{-3x} + c_2 e^{2x} - \frac{5}{13}\cos 2x + \frac{1}{13}\sin 2x.$$

Motivating example

Example

Find the general solution to

$$y'' + y' - 6y = 4e^{2ix}.$$

1. The complementary function is $y_c(x) = c_1 e^{-3x} + c_2 e^{2x}$.
2. If we're using complex numbers, use the trial solution $y_p(x) = c_3 e^{2ix}$.
3. Plugging y_p into the original equation yields $c_3 = -\frac{1}{13}(5+i)$.
4. Thus, the general solution is

$$y(x) = c_1 e^{-3x} + c_2 e^{2x} - \tfrac{1}{13}(5+i)e^{2ix}.$$

Complex-valued trial solutions

Which problem was easier? Depends on your position on complex numbers, but the second only involved one unknown coefficient while the first had two. So it may be advantageous, when the nonhomogeneous term is $cx^k e^{ax} \cos bx$ or $cx^k e^{ax} \sin bx$, to change it to $cx^k e^{(a+bi)x}$, solve, and take the real or imaginary part.

Rigor

Theorem

If $y(x) = u(x) + iv(x)$ *is a complex-valued solution to*

$$P(D)y = F(x) + iG(x),$$

then

$$P(D)u = F(x) \quad \textit{and} \quad P(D)v = G(x).$$

Proof.

If $y(x) = u(x) + iv(x)$, then

$$P(D)y = P(D)(u + iv) = P(D)u + iP(D)v.$$

Equating real and imaginary parts gives

$$P(D)u = F(x) \quad \text{and} \quad P(D)v = G(x).$$

$\mathcal{Q.E.D.}$

Method

Solutions to the nonhomogeneous polynomial differential equations

$$P(D)y = cx^k e^{ax} \cos bx \quad \text{and} \quad P(D)y = cx^k e^{ax} \sin bx,$$

may be found by solving the complex equation

$$P(D)z = cx^k e^{(a+bi)x}$$

and then taking the real and imaginary parts, respectively, of the solution $z(x)$.

Bonus

Solve two equations at once!

Example

Solve $y'' - 2y' + 5y = 8e^x \sin 2x$.

1. The complementary function is $y_c(x) = e^x(c_1 \cos 2x + c_2 \sin 2x)$.
2. Instead, solve $z'' - 2z' + 5z = 8e^{(1+2i)x}$.
3. Since $1 + 2i$ is a root of the auxiliary polynomial, use the trial solution $z_p(x) = c_3 x e^{(1+2i)x}$.
4. Plugging z_p into $z'' - 2z' + 5z = 8e^{(1+2i)x}$ yields $c_3 = -2i$.
5. Thus, the particular solution is
$$z_p(x) = -2ixe^{(1+2i)x} = -2xe^x(-\sin 2x + i \cos 2x).$$
6. To get $8e^x \sin 2x$ on the right-hand side, take the imaginary part
$$y_p(x) = \mathrm{Im}(z_p) = -2xe^x \cos 2x.$$
7. The general solution is
$$y(x) = e^x(c_1 \cos 2x + c_2 \sin 2x) - 2xe^x \cos 2x.$$