Generalized Eigenvectors

Math 240 — Calculus III

Summer 2013, Session II

Wednesday, July 31, 2013

Agenda

1. Definition
2. Computation and Properties
3. Chains

Motivation

Defective matrices cannot be diagonalized because they do not possess enough eigenvectors to make a basis. How can we correct this defect?

Example

The matrix $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ is defective.

1. Only eigenvalue is $\lambda = 1$.
2. $A - I = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$
3. Single eigenvector $\mathbf{v} = (1, 0)$.
4. We could use $\mathbf{u} = (0, 1)$ to complete a basis.
5. Notice that $(A - I)\mathbf{u} = \mathbf{v}$ and $(A - I)^2\mathbf{u} = \mathbf{0}$.

Maybe we just didn't multiply by $A - \lambda I$ enough times.

Definition

Definition

If A is an $n \times n$ matrix, a **generalized eigenvector** of A corresponding to the eigenvalue λ is a nonzero vector $\mathbf{x}$ satisfying

$$(A - \lambda I)^p \mathbf{x} = \mathbf{0}$$

for some positive integer p. Equivalently, it is a nonzero element of the nullspace of $(A - \lambda I)^p$.

Example

- Eigenvectors are generalized eigenvectors with $p = 1$.
- In the previous example we saw that $\mathbf{v} = (1, 0)$ and $\mathbf{u} = (0, 1)$ are generalized eigenvectors for

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \text{ and } \lambda = 1.$$

Computing generalized eigenvectors

Example

Determine generalized eigenvectors for the matrix

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{bmatrix}.$$

1. Characteristic polynomial is $(3-\lambda)(1-\lambda)^2$.
2. Eigenvalues are $\lambda = 1, 3$.
3. Eigenvectors are
$$\lambda_1 = 3: \qquad \mathbf{v}_1 = (1,2,2),$$
$$\lambda_2 = 1: \qquad \mathbf{v}_2 = (1,0,0).$$
4. Final generalized eigenvector will a vector $\mathbf{v}_3 \neq \mathbf{0}$ such that
$$(A - \lambda_2 I)^2 \mathbf{v}_3 = \mathbf{0} \text{ but } (A - \lambda_2 I)\mathbf{v}_3 \neq \mathbf{0}.$$
Pick $\mathbf{v}_3 = (0,1,0)$. Note that $(A-\lambda_2 I)\mathbf{v}_3 = \mathbf{v}_2$.

Facts about generalized eigenvectors

How many powers of $(A - \lambda I)$ do we need to compute in order to find all of the generalized eigenvectors for λ?

Fact

If A is an $n \times n$ matrix and λ is an eigenvalue with algebraic multiplicity k, then the set of generalized eigenvectors for λ consists of the nonzero elements of $\mathrm{nullspace}\left((A - \lambda I)^k\right)$.

In other words, we need to take at most k powers of $A - \lambda I$ to find all of the generalized eigenvectors for λ.

Computing generalized eigenvectors

Example

Determine generalized eigenvectors for the matrix

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 1 & 2 \\ 0 & -1 & 1 \end{bmatrix}.$$

1. Single eigenvalue of $\lambda = 1$.
2. Single eigenvector $\mathbf{v}_1 = (-2, 0, 1)$.
3. Look at

   $$(A - I)^2 = \begin{bmatrix} 2 & 0 & 4 \\ 0 & 0 & 0 \\ -1 & 0 & -2 \end{bmatrix}$$

   to find generalized eigenvector $\mathbf{v}_2 = (0, 1, 0)$.
4. Finally, $(A - I)^3 = \mathbf{0}$, so we get $\mathbf{v}_3 = (1, 0, 0)$.

Facts about generalized eigenvectors

The aim of generalized eigenvectors was to enlarge a set of linearly independent eigenvectors to make a basis. Are there always enough generalized eigenvectors to do so?

Fact

If λ is an eigenvalue of A with algebraic multiplicity k, then

$$\operatorname{nullity}\left((A-\lambda I)^k\right) = k.$$

In other words, there are k linearly independent generalized eigenvectors for λ.

Corollary

If A is an $n \times n$ matrix, then there is a basis for $\mathbb{R}^n$ consisting of generalized eigenvectors of A.

Computing generalized eigenvectors

Example

Determine generalized eigenvectors for the matrix

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 1 & 2 \\ 0 & -1 & 1 \end{bmatrix}.$$

1. From last time, we have eigenvalue $\lambda = 1$ and eigenvector $\mathbf{v}_1 = (-2, 0, 1)$.
2. Solve $(A - I)\mathbf{v}_2 = \mathbf{v}_1$ to get $\mathbf{v}_2 = (0, -1, 0)$.
3. Solve $(A - I)\mathbf{v}_3 = \mathbf{v}_2$ to get $\mathbf{v}_3 = (-1, 0, 0)$.

Chains of generalized eigenvectors

Let A be an $n \times n$ matrix and $\mathbf{v}$ a generalized eigenvector of A corresponding to the eigenvalue λ. This means that

$$(A - \lambda I)^p \, \mathbf{v} = \mathbf{0}$$

for a positive integer p.

If $0 \leq q < p$, then

$$(A - \lambda I)^{p-q} \, (A - \lambda I)^q \, \mathbf{v} = \mathbf{0}.$$

That is, $(A - \lambda I)^q \mathbf{v}$ is also a generalized eigenvector corresponding to λ for $q = 0, 1, \ldots, p-1$.

Definition

If p is the smallest positive integer such that $(A - \lambda I)^p \, \mathbf{v} = \mathbf{0}$, then the sequence

$$(A - \lambda I)^{p-1} \, \mathbf{v}, \;\; (A - \lambda I)^{p-2} \, \mathbf{v}, \;\; \ldots, \;\; (A - \lambda I) \, \mathbf{v}, \;\; \mathbf{v}$$

is called a **chain** or **cycle** of generalized eigenvectors. The integer p is called the **length** of the cycle.

Chains of generalized eigenvectors

Example

In the previous example,

$$A - \lambda I = \begin{bmatrix} 0 & 2 & 0 \\ 1 & 0 & 2 \\ 0 & -1 & 0 \end{bmatrix}$$

and we found the chain

$$\mathbf{v} = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}, \ (A - \lambda I)\mathbf{v} = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}, \ (A - \lambda I)^2\mathbf{v} = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}.$$

Fact

The generalized eigenvectors in a chain are linearly independent.

Jordan canonical form

What's the analogue of diagonalization for defective matrices? That is, if $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ are the linearly independent generalized eigenvectors of A, what does the matrix $S^{-1}AS$ look like, where $S = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \end{bmatrix}$?