Vector Differential Equations: Nondefective Coefficient Matrix

Math 240 — Calculus III

Summer 2013, Session II

Tuesday, July 30, 2013

Agenda

1. Solving linear systems by diagonalization
   - Real eigenvalues
   - Complex eigenvalues

Introduction

The results discussed yesterday apply to any old vector differential equation

$$\mathbf{x}' = A\mathbf{x}.$$

In order to make some headway in solving them, however, we must make a simplifying assumption:

The coefficient matrix A consists of real *constants*.

Diagonalization

Recall that an $n \times n$ matrix A may be diagonalized if and only if it is nondefective.

When this happens, we can solve the homogeneous vector differential equation

$$\mathbf{x}' = A\mathbf{x}.$$

If $S^{-1}AS = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$, then

$$\mathbf{x} = S\mathbf{y}, \text{ where } \mathbf{y} = \begin{bmatrix} c_1 e^{\lambda_1 t} \\ c_2 e^{\lambda_2 t} \\ \vdots \\ c_n e^{\lambda_n t} \end{bmatrix}.$$

Example

Solve the linear system

$$\begin{aligned} x_1' &= 2x_1 + x_2, \\ x_2' &= -3x_1 - 2x_2. \end{aligned}$$

1. Turn it into the vector differential equation
$$\mathbf{x}' = A\mathbf{x}, \quad \text{where } A = \begin{bmatrix} 2 & 1 \\ -3 & -2 \end{bmatrix}.$$
2. The characteristic polynomial of A is $\lambda^2 - 1$.
3. Eigenvalues are $\lambda = \pm 1$.
4. Eigenvectors are
$$\begin{aligned} \lambda_1 = 1 : &\qquad \mathbf{v}_1 = (-1, 1), \\ \lambda_2 = -1 : &\qquad \mathbf{v}_2 = (-1, 3). \end{aligned}$$
5. We have
$$\mathbf{y} = \begin{bmatrix} c_1e^t \\ c_2e^{-t} \end{bmatrix}, \text{ so } \mathbf{x} = \begin{bmatrix} -1 & -1 \\ 1 & 3 \end{bmatrix} \mathbf{y} = \begin{bmatrix} -c_1e^t - c_2e^{-t} \\ c_1e^t + 3c_2e^{-t} \end{bmatrix}.$$

Vector formulation

The change of basis matrix S is

$$S = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \end{bmatrix},$$

where $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are n linearly independent eigenvectors of A. Hence,

$$\begin{aligned} \mathbf{x} = S\mathbf{y} &= c_1 e^{\lambda_1 t}\mathbf{v}_1 + c_2 e^{\lambda_2 t}\mathbf{v}_2 + \cdots + c_n e^{\lambda_n t}\mathbf{v}_n \\ &= c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_n\mathbf{x}_n. \end{aligned}$$

Check if these n solutions are linearly independent:

$$\begin{aligned} W[\mathbf{x}_1, \ldots, \mathbf{x}_n] &= \det\left(\begin{bmatrix} e^{\lambda_1 t}\mathbf{v}_1 & e^{\lambda_2 t}\mathbf{v}_2 & \cdots & e^{\lambda_n t}\mathbf{v}_n \end{bmatrix}\right) \\ &= e^{(\lambda_1 + \cdots + \lambda_n)t} \det\left(\begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \end{bmatrix}\right) \\ &\neq 0. \end{aligned}$$

They are linearly independent, therefore a fundamental set of solutions.

General solution

Theorem

Suppose A is an $n \times n$ matrix of real constants. If A has n real linearly independent eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ with corresponding eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ (not necessarily distinct), then the vector functions $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n\}$ defined by

$$\mathbf{x}_k(t) = e^{\lambda_k t}\mathbf{v}_k, \quad \text{for } k = 1, 2, \ldots, n$$

are a fundamental set of solutions to $\mathbf{x}' = A\mathbf{x}$ on any interval. The general solution is

$$\mathbf{x}(t) = c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_n\mathbf{x}_n.$$

Example

Find the general solution to $\mathbf{x}' = A\mathbf{x}$ if

$$A = \begin{bmatrix} 0 & 2 & -3 \\ -2 & 4 & -3 \\ -2 & 2 & -1 \end{bmatrix}.$$

1. Characteristic polynomial is $-(\lambda+1)(\lambda-2)^2$.
2. Eigenvalues are $\lambda_1 = -1$ and $\lambda_2 = 2$.
3. Eigenvectors are

$$\begin{aligned} \lambda_1 = -1 : &\qquad \mathbf{v}_1 = (1,1,1), \\ \lambda_2 = 2 : &\qquad \mathbf{v}_2 = (1,1,0), \qquad \mathbf{v}_3 = (-3,0,2). \end{aligned}$$

4. Fundamental set of solution is

$$\mathbf{x}_1(t) = e^{-t}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \ \mathbf{x}_2(t) = e^{2t}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \ \mathbf{x}_3(t) = e^{2t}\begin{bmatrix} -3 \\ 0 \\ 2 \end{bmatrix}.$$

5. So general solution is

$$\mathbf{x}(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) + c_3\mathbf{x}_3(t).$$

Complex eigenvalues

What happens when A has complex eigenvalues?

If $u = a + ib$ and $v = a - ib$ then

$$a = \frac{u+v}{2} \quad \text{and} \quad b = \frac{u-v}{2i}.$$

Theorem

Let $\mathbf{u}(t)$ and $\mathbf{v}(t)$ be real-valued vector functions. If

$$\mathbf{w}_1(t) = \mathbf{u}(t) + i\mathbf{v}(t) \quad \textit{and} \quad \mathbf{w}_2(t) = \mathbf{u}(t) - i\mathbf{v}(t)$$

are complex conjugate solutions to $\mathbf{x}' = A\mathbf{x}$, then

$$\mathbf{x}_1(t) = \mathbf{u}(t) \quad \textit{and} \quad \mathbf{x}_2(t) = \mathbf{v}(t)$$

are themselves <u>real valued</u> solutions of $\mathbf{x}' = A\mathbf{x}$.

Complex eigenvalue example

Example

Find the general solution to $\mathbf{x}' = A\mathbf{x}$ where

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

1. Characteristic polynomial is $\lambda^2 + 1$.
2. Eigenvalues are $\lambda = \pm i$.
3. Eigenvectors are $\mathbf{v} = (1, \pm i)$.
4. Linearly independent solutions are
$$\mathbf{w}(t) = e^{\pm it} \begin{bmatrix} 1 \\ \pm i \end{bmatrix} = \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix} \pm i \begin{bmatrix} \sin t \\ \cos t \end{bmatrix}.$$
5. Yields the two linearly independent real solutions
$$\mathbf{x}_1(t) = \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix} \quad \text{and} \quad \mathbf{x}_2(t) = \begin{bmatrix} \sin t \\ \cos t \end{bmatrix}.$$

The formula

Let's derive the explicit form of the real solutions produced by a pair of complex conjugate eigenvectors.

Suppose $\lambda = a + ib$ is an eigenvalue of A, with $b \neq 0$, corresponding to the eigenvector $\mathbf{r} + i\mathbf{s}$. This produces the complex solution

$$\begin{aligned}\mathbf{w}(t) &= e^{(a+ib)t}(\mathbf{r} + i\mathbf{s}) \\ &= e^{at}(\cos bt + i \sin bt)(\mathbf{r} + i\mathbf{s}) \\ &= e^{at}(\cos bt\,\mathbf{r} - \sin bt\,\mathbf{s}) + ie^{at}(\sin bt\,\mathbf{r} + \cos bt\,\mathbf{s}).\end{aligned}$$

Thus, the two real-valued solutions to $\mathbf{x}' = A\mathbf{x}$ are

$$\begin{aligned}\mathbf{x}_1(t) &= e^{at}(\cos bt\,\mathbf{r} - \sin bt\,\mathbf{s}), \\ \mathbf{x}_2(t) &= e^{at}(\sin bt\,\mathbf{r} + \cos bt\,\mathbf{s}).\end{aligned}$$

Remark

The conjugate eigenvalue $a - ib$ and eigenvector $\mathbf{r} - i\mathbf{s}$ would result in the same pair of real solutions.