Linear Systems of Differential Equations

Math 240 — Calculus III

Summer 2013, Session II

Monday, July 29, 2013

Agenda

1. First order linear systems
   - Solutions to vector differential equations
   - Beyond first order systems

First order linear systems

Definition

A **first order system of differential equations** is of the form

$$\mathbf{x}'(t) = A(t)\mathbf{x}(t) + \mathbf{b}(t),$$

where $A(t)$ is an $n \times n$ matrix function and $\mathbf{x}(t)$ and $\mathbf{b}(t)$ are n-vector functions. Also called a **vector differential equation**.

Example

The linear system

$$\begin{aligned} x_1'(t) &= \cos(t)x_1(t) - \sin(t)x_2(t) + e^{-t} \\ x_2'(t) &= \sin(t)x_1(t) + \cos(t)x_2(t) - e^{-t} \end{aligned}$$

can also be written as the vector differential equation

$$\mathbf{x}'(t) = A(t)\mathbf{x}(t) + \mathbf{b}(t)$$

where

$$A(t) = \begin{bmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{bmatrix}, \quad \mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \quad \text{and } \mathbf{b}(t) = \begin{bmatrix} e^{-t} \\ -e^{-t} \end{bmatrix}.$$

The vector space $V_n(I)$

A **solution** to a vector differential equation will be an element of the vector space $V_n(I)$ consisting of column n-vector functions defined on the interval I.

Definition

Suppose $\mathbf{x}_1(t), \mathbf{x}_2(t), \ldots, \mathbf{x}_n(t) \in V_n(I)$. The **Wronskian** of these vectors is

$$W[\mathbf{x}_1, \ldots, \mathbf{x}_n](t) = \begin{vmatrix} | & | & & | \\ \mathbf{x}_1(t) & \mathbf{x}_2(t) & \cdots & \mathbf{x}_n(t) \\ | & | & & | \end{vmatrix}.$$

Theorem

If $W[\mathbf{x}_1, \ldots, \mathbf{x}_n](t)$ is nonzero for at least one $t \in I$, then $\{\mathbf{x}_1(t), \ldots, \mathbf{x}_n(t)\}$ is a linearly independent subset of $V_n(I)$.

Solutions to homogeneous linear systems

As with linear systems, a homogeneous linear system of differential equations is one in which $\mathbf{b}(t) = 0$.

Theorem

If $A(t)$ is an $n \times n$ matrix function that is continuous on the interval I, then the set of all solutions to $\mathbf{x}'(t) = A(t)\mathbf{x}(t)$ is a subspace of $V_n(I)$ of dimension n.

Proof.

Up to you. Proof of $\dim = n$ later, if there's time. $\mathcal{Q.E.D.}$

The general solution: homogeneous case

If the solution set is a vector space of dimension n, it has a basis.

Definition

Any set $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n\}$ of n solutions to $\mathbf{x}' = A\mathbf{x}$ that is linearly independent on I is called a **fundamental set of solutions** on I. Any solution may be written in the form

$$\mathbf{x}(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) + \cdots + c_n\mathbf{x}_n(t),$$

which is called the **general solution**.

Theorem

If $A(t)$ is an $n \times n$ matrix function that is continuous on an interval I, and $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n\}$ is a linearly independent set of solutions to $\mathbf{x}' = A\mathbf{x}$ on I, then

$$W[\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n](t) \neq 0$$

for every $t \in I$.

The general solution: nonhomogeneous case

The case of nonhomogeneous systems is also familiar.

Theorem

Suppose $A(t)$ is an $n \times n$ matrix function continuous on an interval I and $\{\mathbf{x}_1, \dots, \mathbf{x}_n\}$ is a fundamental set of solutions to the equation $\mathbf{x}' = A\mathbf{x}$. If $\mathbf{x} = \mathbf{x}_p(t)$ is any particular solution to the nonhomogeneous vector differential equation

$$\mathbf{x}'(t) = A(t)\mathbf{x}(t) + \mathbf{b}(t)$$

on I, then every solution to this equation on I is in the form of the **general solution**

$$\begin{aligned} \mathbf{x}'(t) &= \underbrace{c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) + \cdots + c_n\mathbf{x}_n(t)}_{\mathbf{x}_c(t)} + \mathbf{x}_p(t), \\ &= \mathbf{x}_c(t) + \mathbf{x}_p(t) \end{aligned}$$

where $\mathbf{x}_p(t)$ is any particular solution.

The two pieces of the general solution are the **particular solution**, $\mathbf{x}_p(t)$, and the **complementary solution**, $\mathbf{x}_c(t)$.

Initial value problems

Sometimes, we are interested in one particular solution to a vector differential equation.

Definition

An **initial value problem** consists of a vector differential equation

$$\mathbf{x}'(t) = A(t)\mathbf{x}(t) + \mathbf{b}(t)$$

and an **initial condition**

$$\mathbf{x}(t_0) = \mathbf{x}_0$$

with known, fixed values for $t_0 \in \mathbb{R}$ and $\mathbf{x}_0 \in \mathbb{R}^n$.

Theorem

When $A(t)$ and $\mathbf{b}(t)$ are continuous on an interval I, the above initial value problem has a unique solution on I.

Turning higher order linear systems into first order

Aren't we a little limited if all we can solve are first order differential equations? No.

Example

Consider the linear *second* order system

$$\begin{aligned} x''(t) - \ 4y(t) &= e^t, \\ y''(t) + t^2 x'(t) &= \sin t. \end{aligned}$$

Introduce new variables

$$x_1(t) = x(t), \quad x_2(t) = x'(t), \quad x_3(t) = y(t), \quad x_4(t) = y'(t).$$

Then the above equations can be replaced with

$$\begin{aligned} x_2'(t) - \ 4x_3(t) &= e^t, \\ x_4'(t) + t^2 x_2(t) &= \sin t, \end{aligned}$$

and we must supplement them with the equations

$$x_1'(t) = x_2(t), \quad x_3'(t) = x_4(t).$$