Eigenvalues, Eigenvectors, and Diagonalization

Math 240 — Calculus III

Summer 2013, Session II

Wednesday, July 24, 2013

Agenda

1. Eigenvalues and Eigenvectors
2. Diagonalization

Introduction

Next week, we will apply linear algebra to solving differential equations. One that is particularly easy to solve is

$$y' = ay.$$

It has the solution $y = ce^{at}$, where c is any real (or complex) number. Viewed in terms of linear transformations, $y = ce^{at}$ is the solution to the vector equation

$$T(y) = ay, \tag{1}$$

where $T : C^k(I) \to C^{k-1}(I)$ is $T(y) = y'$. We are going to study equation (1) in a more general context.

Definition

Let A be an $n \times n$ matrix. Any value of λ for which

$$A\mathbf{v} = \lambda \mathbf{v}$$

has *nontrivial* solutions $\mathbf{v}$ are called **eigenvalues** of A. The corresponding *nonzero* vectors $\mathbf{v}$ are called **eigenvectors** of A.

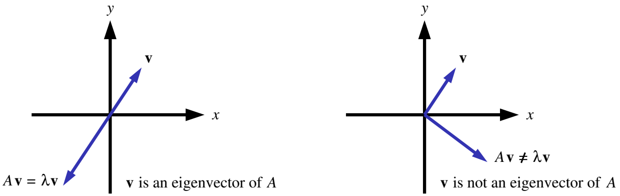

Figure: A geometrical description of eigenvectors in $\mathbb{R}^2$.

Example

Example

If A is the matrix

$$A = \begin{bmatrix} 1 & 1 \\ -3 & 5 \end{bmatrix},$$

then the vector $\mathbf{v} = (1, 3)$ is an eigenvector for A because

$$A\mathbf{v} = \begin{bmatrix} 1 & 1 \\ -3 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ 12 \end{bmatrix} = 4\mathbf{v}.$$

The corresponding eigenvalue is $\lambda = 4$.

Remark

Note that if $A\mathbf{v} = \lambda\mathbf{v}$ and c is any scalar, then

$$A(c\mathbf{v}) = c\,A\mathbf{v} = c(\lambda\mathbf{v}) = \lambda(c\mathbf{v}).$$

Consequently, if $\mathbf{v}$ is an eigenvector of A, then so is $c\mathbf{v}$ for any nonzero scalar c.

Finding eigenvalues

The eigenvector/eigenvalue equation can be rewritten as

$$(A - \lambda I)\,\mathbf{v} = \mathbf{0}.$$

The eigenvalues of A are the values of λ for which the above equation has nontrivial solutions. There are nontrivial solutions if and only if

$$\det(A - \lambda I) = 0.$$

Definition

For a given $n \times n$ matrix A, the polynomial

$$p(\lambda) = \det(A - \lambda I)$$

is called the **characteristic polynomial** of A, and the equation

$$p(\lambda) = 0$$

is called the **characteristic equation** of A.

The eigenvalues of A are the roots of its characteristic polynomial.

Finding eigenvectors

If λ is a root of the characteristic polynomial, then the nonzero elements of

$$\operatorname{nullspace}(A - \lambda I)$$

will be eigenvectors for A.

Since nonzero linear combinations of eigenvectors for a single eigenvalue are still eigenvectors, we'll find a set of linearly independent eigenvectors for each eigenvalue.

Example

Find all of the eigenvalues and eigenvectors of

$$A = \begin{bmatrix} 5 & -4 \\ 8 & -7 \end{bmatrix}.$$

Compute the characteristic polynomial

$$\det(A - \lambda I) = \begin{vmatrix} 5-\lambda & -4 \\ 8 & -7-\lambda \end{vmatrix} = \lambda^2 + 2\lambda - 3.$$

Its roots are $\lambda = -3$ and $\lambda = 1$. These are the eigenvalues.
If $\lambda = -3$, we have the eigenvector $(1, 2)$.
If $\lambda = 1$, then

$$A - I = \begin{bmatrix} 4 & -4 \\ 8 & -8 \end{bmatrix},$$

which gives us the eigenvector $(1, 1)$.

Repeated eigenvalues

Find all of the eigenvalues and eigenvectors of

$$A = \begin{bmatrix} 5 & 12 & -6 \\ -3 & -10 & 6 \\ -3 & -12 & 8 \end{bmatrix}.$$

Compute the characteristic polynomial $-(\lambda - 2)^2(\lambda + 1)$.

Definition

If A is a matrix with characteristic polynomial $p(\lambda)$, the multiplicity of a root λ of p is called the **algebraic multiplicity** of the eigenvalue λ.

Example

In the example above, the eigenvalue $\lambda = 2$ has algebraic multiplicity 2, while $\lambda = -1$ has algebraic multiplicity 1.

Repeated eigenvalues

The eigenvalue $\lambda = 2$ gives us two linearly independent eigenvectors $(-4, 1, 0)$ and $(2, 0, 1)$.
When $\lambda = -1$, we obtain the single eigenvector $(-1, 1, 1)$.

Definition

The number of linearly independent eigenvectors corresponding to a single eigenvalue is its **geometric multiplicity**.

Example

Above, the eigenvalue $\lambda = 2$ has geometric multiplicity 2, while $\lambda = -1$ has geometric multiplicity 1.

Theorem

The geometric multiplicity of an eigenvalue is less than or equal to its algebraic multiplicity.

Definition

A matrix that has an eigenvalue whose geometric multiplicity is less than its algebraic multiplicity is called **defective**.

A defective matrix

Find all of the eigenvalues and eigenvectors of

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

The characteristic polynomial is $(\lambda - 1)^2$, so we have a single eigenvalue $\lambda = 1$ with algebraic multiplicity 2.
The matrix

$$A - I = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

has a one-dimensional null space spanned by the vector $(1, 0)$. Thus, the geometric multiplicity of this eigenvalue is 1.

Complex eigenvalues

Find all of the eigenvalues and eigenvectors of

$$A = \begin{bmatrix} -2 & -6 \\ 3 & 4 \end{bmatrix}.$$

The characteristic polynomial is $\lambda^2 - 2\lambda + 10$. Its roots are

$$\lambda_1 = 1 + 3i \quad \text{and} \quad \lambda_2 = \overline{\lambda_1} = 1 - 3i.$$

The eigenvector corresponding to λ_1 is $(-1 + i, 1)$.

Theorem

Let A be a square matrix with real elements. If λ is a complex eigenvalue of A with eigenvector $\mathbf{v}$, then $\overline{\lambda}$ is an eigenvalue of A with eigenvector $\overline{\mathbf{v}}$.

Example

The eigenvector corresponding to $\lambda_2 = \overline{\lambda_1}$ is $(-1 - i, 1)$.

Segue

If an $n \times n$ matrix A is nondefective, then a set of linearly independent eigenvectors for A will form a basis for $\mathbb{R}^n$. If we express the linear transformation $T(\mathbf{x}) = A\mathbf{x}$ as a matrix transformation relative to this basis, it will look like

$$\begin{bmatrix} \lambda_1 & & & \Large{0} \\ & \lambda_2 & & \\ & & \ddots & \\ \Large{0} & & & \lambda_n \end{bmatrix}.$$

The following example will demonstrate the utility of such a representation.

Differential equation example

Determine all solutions to the linear system of differential equations

$$\mathbf{x}' = \begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 5x_1 - 4x_2 \\ 8x_1 - 7x_2 \end{bmatrix} = \begin{bmatrix} 5 & -4 \\ 8 & -7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = A\mathbf{x}.$$

We know that the coefficient matrix has eigenvalues $\lambda_1 = 1$ and $\lambda_2 = -3$ with corresponding eigenvectors $\mathbf{v}_1 = (1,1)$ and $\mathbf{v}_2 = (1,2)$, respectively. Using the basis $\{\mathbf{v}_1, \mathbf{v}_2\}$, we write the linear transformation $T(\mathbf{x}) = A\mathbf{x}$ in the matrix representation

$$\begin{bmatrix} 1 & 0 \\ 0 & -3 \end{bmatrix}.$$

Differential equation example

Now consider the new linear system

$$\mathbf{y}' = \begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = B\mathbf{y}.$$

It has the obvious solution

$$y_1 = c_1 e^t \quad \text{and} \quad y_2 = c_2 e^{-3t},$$

for any scalars c_1 and c_2. How is this relevant to $\mathbf{x}' = A\mathbf{x}$?

$$A\begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix} = \begin{bmatrix} A\mathbf{v}_1 & A\mathbf{v}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{v}_1 & -3\mathbf{v}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix} B.$$

Let $S = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix}$. Since $\mathbf{y}' = B\mathbf{y}$ and $AS = SB$, we have

$$(S\mathbf{y})' = S\mathbf{y}' = SB\mathbf{y} = AS\mathbf{y} = A\,(S\mathbf{y})\,.$$

Thus, a solution to $\mathbf{x}' = A\mathbf{x}$ is given by

$$\mathbf{x} = S\mathbf{y} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} c_1 e^t \\ c_2 e^{-3t} \end{bmatrix} = \begin{bmatrix} c_1 e^t + c_2 e^{-3t} \\ c_1 e^t + 2c_2 e^{-3t} \end{bmatrix}.$$