Solving Linear Systems, Continued
and
The Inverse of a Matrix

Math 240 — Calculus III

Summer 2013, Session II

Monday, July 15, 2013

Agenda

Gaussian elimination

Gaussian elimination solves a linear system by reducing to REF via elementary row ops and then using back substitution.

Example

$$\begin{array}{rcrcrcr} 3x_1 & - & 2x_2 & + & 2x_3 & = & 9 \\ x_1 & - & 2x_2 & + & x_3 & = & 5 \\ 2x_1 & - & x_2 & - & 2x_3 & = & -1 \end{array} \rightsquigarrow \begin{bmatrix} 3 & -2 & 2 & 9 \\ 1 & -2 & 1 & 5 \\ 2 & -1 & -2 & -1 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & -2 & 1 & 5 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & 2 \end{bmatrix} \rightsquigarrow \begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & x_3 & = & 5 \\ & & x_2 & + & 3x_3 & = & 5 \\ & & & & x_3 & = & 2 \end{array}$$

Steps

1. P_{12}
2. $A_{12}(-3)$
3. $A_{13}(-2)$
4. $A_{32}(-1)$
5. $A_{23}(-3)$
6. $M_3\left(\frac{-1}{13}\right)$

Back substitution gives the solution $(1, -1, 2)$.

Gauss-Jordan elimination

Reducing the augmented matrix to RREF makes the system even easier to solve.

Example

$$\left[\begin{array}{rrr|r} 1 & -2 & 1 & 5 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & 2 \end{array}\right] \to \left[\begin{array}{rrr|r} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{array}\right] \rightsquigarrow \begin{array}{rcrcrcr} x_1 & & & & & = & 1 \\ & & x_2 & & & = & -1 \\ & & & & x_3 & = & 2 \end{array}$$

Steps

1. $A_{32}(-3)$
2. $A_{31}(-1)$
3. $A_{21}(2)$

Now, without any back substitution, we can see that the solution is $(1, -1, 2)$.

The method of solving a linear system by reducing its augmented matrix to RREF is called **Gauss-Jordan elimination**.

The rank of a matrix

Definition

The **rank** of a matrix, A, is the number of nonzero rows it has after reduction to REF. It is denoted by $\operatorname{rank}(A)$.

If A the coefficient matrix of an $m \times n$ linear system and $\operatorname{rank}(A^\#) = \operatorname{rank}(A) = n$ then the REF looks like

$$\begin{bmatrix} 1 & * & * & \cdots & * \\ & 1 & * & \dots & * \\ & & \ddots & & \vdots \\ \text{\Huge 0} & & & 1 & * \\ 0 & \dots & \dots & \dots & 0 \end{bmatrix} \rightsquigarrow \begin{matrix} x_1 & = & * \\ x_2 & = & * \\ & \vdots & \\ x_n & = & * \end{matrix}$$

Lemma

Suppose $A\mathbf{x} = \mathbf{b}$ *is an* $m \times n$ *linear system with augmented matrix* $A^\#$*. If* $\operatorname{rank}(A^\#) = \operatorname{rank}(A) = n$ *then the system has a unique solution.*

The rank of a matrix

Example

Determine the solution set of the linear system

$$\begin{aligned} x_1 + x_2 - x_3 + x_4 &= 1,\\ 2x_1 + 3x_2 + x_3 &= 4,\\ 3x_1 + 5x_2 + 3x_3 - x_4 &= 5. \end{aligned}$$

Reduce the augmented matrix.

$$\begin{bmatrix} 1 & 1 & -1 & 1 & 1 \\ 2 & 3 & 1 & 0 & 4 \\ 3 & 5 & 3 & -1 & 5 \end{bmatrix} \xrightarrow[A_{23}(-2)]{\substack{A_{12}(-2)\\ A_{13}(-3)}} \begin{bmatrix} 1 & 1 & -1 & 1 & 1 \\ 0 & 1 & 3 & -2 & 2 \\ 0 & 0 & 0 & 0 & -2 \end{bmatrix}$$

The last row says $0 = -2$; the system is inconsistent.

Lemma

Suppose $A\mathbf{x} = \mathbf{b}$ *is a linear system with augmented matrix* $A^{\#}$*. If* $\operatorname{rank}(A^{\#}) > \operatorname{rank}(A)$ *then the system is inconsistent.*

The rank of a matrix

Example

Determine the solution set of the linear system

$$\begin{aligned} 5x_1 - 6x_2 + x_3 &= 4,\\ 2x_1 - 3x_2 + x_3 &= 1,\\ 4x_1 - 3x_2 - x_3 &= 5. \end{aligned}$$

Reduce the augmented matrix.

$$\begin{bmatrix}5 & -6 & 1 & 4\\ 2 & -3 & 1 & 1\\ 4 & -3 & -1 & 5\end{bmatrix} \to \begin{bmatrix}1 & 0 & -1 & 2\\ 0 & 1 & -1 & 1\\ 0 & 0 & 0 & 0\end{bmatrix} \rightsquigarrow \begin{array}{rrcrcl} x_1 & & - & x_3 & = & 2\\ & x_2 & - & x_3 & = & 1 \end{array}$$

The unknown x_3 can assume any value. Let $x_3 = t$. Then by back substitution we get $x_2 = t + 1$ and $x_1 = t + 2$. Thus, the solution set is the line

$$\{(t+2, t+1, t) : t \in \mathbb{R}\}.$$

The rank of a matrix

Definition

When an unknown variable in a linear system is free to assume any value, we call it a **free variable**. Variables that are not free are called **bound variables**.

The value of a bound variable is uniquely determined by a choice of values for all of the free variables in the system.

Lemma

Suppose $A\mathbf{x} = \mathbf{b}$ *is an* $m \times n$ *linear system with augmented matrix* $A^{\#}$*. If* $\operatorname{rank}(A^{\#}) = \operatorname{rank}(A) < n$ *then the system has an infinite number of solutions. Such a system will have* $n - \operatorname{rank}(A^{\#})$ *free variables.*

Solving linear systems with free variables

Example

Use Gaussian elimination to solve

$$\begin{aligned} x_1 + 2x_2 - 2x_3 - x_4 &= 3, \\ 3x_1 + 6x_2 + x_3 + 11x_4 &= 16, \\ 2x_1 + 4x_2 - x_3 + 4x_4 &= 9. \end{aligned}$$

Reducing to row-echelon form yields

$$\begin{aligned} x_1 + 2x_2 - 2x_3 - x_4 &= 3, \\ x_3 + 2x_4 &= 1. \end{aligned}$$

Choose as free variables those variables that **do not** have a pivot in their column.

In this case, our free variables will be x_2 and x_4. The solution set is the plane

$$\{(5 - 2s - 3t, s, 1 - 2t, t) : s, t \in \mathbb{R}\}.$$

The inverse of a square matrix

Can we divide by a matrix? What properties should the inverse matrix have?

Definition

Suppose A is a square, $n \times n$ matrix. An **inverse matrix** for A is an $n \times n$ matrix, B, such that

$$AB = I_n \quad \text{and} \quad BA = I_n.$$

If A has such an inverse then we say that it is **invertible** or **nonsingular**. Otherwise, we say that A is **singular**.

Remark

Not every matrix is invertible.

If you have a linear system $A\mathbf{x} = \mathbf{b}$ and B is an inverse matrix for A then the linear system has the unique solution

$$\mathbf{x} = B\mathbf{b}.$$

The inverse of a square matrix

Example

If

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 2 & -3 & 3 \\ 1 & -1 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & -1 & 3 \\ 1 & -1 & 1 \\ 1 & 0 & -1 \end{bmatrix} = A^{-1}$$

then B is *the* inverse of A.

Theorem (Matrix inverses are well-defined)

Suppose A is an $n \times n$ matrix. If B and C are two inverses of A then $B = C$.

Thus, we can write A^{-1} for *the* inverse of A with no ambiguity.

Useful Example

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $ad - bc \neq 0$ then $A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

Finding the inverse of a matrix

Inverse matrices sound great! How do I find one?
Suppose A is a 3×3 invertible matrix. If $A^{-1} = \begin{bmatrix} \mathbf{x}_1 & \mathbf{x}_2 & \mathbf{x}_3 \end{bmatrix}$ then

$$A\mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \; A\mathbf{x}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \text{ and } A\mathbf{x}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

We can find A^{-1} by solving 3 linear systems at once!

In general, form the augmented matrix and reduce to RREF. You end up with A^{-1} on the right.

$$\left[A \,\middle|\, I_n \right] \rightsquigarrow \left[I_n \,\middle|\, A^{-1} \right]$$

Finding the inverse of a matrix

Example

Let's find the inverse of $A = \begin{bmatrix} 1 & -1 & 2 \\ 2 & -3 & 3 \\ 1 & -1 & 1 \end{bmatrix}$.

Take the augmented matrix and row reduce.

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 2 & -3 & 3 & 0 & 1 & 0 \\ 1 & -1 & 1 & 0 & 0 & 1 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & -1 & 3 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 1 & 0 & -1 \end{array}\right]$$

$$\underbrace{\begin{matrix} 0 & -1 & 3 \\ 1 & -1 & 1 \\ 1 & 0 & -1 \end{matrix}}_{A^{-1}}$$

Steps

1. $A_{12}(-2)$
2. $A_{13}(-1)$
3. $M_2(-1)$
4. $M_3(-1)$
5. $A_{32}(-1)$
6. $A_{31}(-2)$
7. $A_{21}(1)$

Finding the inverse of a matrix

In order to find the inverse of a matrix, A, we row reduced an augmented matrix with A on the left. What if we don't end up with I_n on the left?

Theorem

An $n \times n$ matrix, A, is invertible if and only if $\operatorname{rank}(A) = n$.

Example

Find the inverse of the matrix $A = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$.

Try to reduce the matrix to RREF.

$$\begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} \xrightarrow{A_{12}(-2)} \begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix}$$

Since $\operatorname{rank}(A) < 2$, we conclude that A is not invertible.
Notice that $(1)(6) - (3)(2) = 0$.

Finding the inverse of a matrix

Diagonal matrices have simple inverses.

Proposition

The inverse of a diagonal matrix is the diagonal matrix with reciprocal entries.

$$\begin{bmatrix} a_{11} & & 0 \\ & \ddots & \\ 0 & & a_{nn} \end{bmatrix}^{-1} = \begin{bmatrix} a_{11}^{-1} & & 0 \\ & \ddots & \\ 0 & & a_{nn}^{-1} \end{bmatrix}$$

Upper and lower triangular matrices have inverses of the same form.

Proposition

The inverse of an upper triangular matrix is upper triangular.
The inverse of a lower triangular matrix is lower triangular.

Properties of inverse matrices

Suppose A and B are $n \times n$ invertible matrices.

- A^{-1} is invertible and $\left(A^{-1}\right)^{-1} = A$.
- AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$.
- A^T is invertible and $\left(A^T\right)^{-1} = \left(A^{-1}\right)^T$.

Corollary

Suppose A_1, A_2, …, A_k *are invertible* $n \times n$ *matrices. Then their product,* $A_1A_2\cdots A_k$ *is invertible, and*

$$(A_1A_2\cdots A_k)^{-1} = A_k^{-1}A_{k-1}^{-1}\cdots A_1^{-1}.$$

Using inverse matrices

Recall that if A is an invertible matrix then the linear system $A\mathbf{x} = \mathbf{b}$ has the unique solution $\mathbf{x} = A^{-1}\mathbf{b}$.

Example

Solve the linear system

$$\begin{array}{rcrcl} x_1 & + & 3x_2 & = & 1, \\ 2x_1 & + & 5x_2 & = & 3. \end{array}$$

The coefficient matrix is $A = \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}$, so $A^{-1} = \begin{bmatrix} -5 & 3 \\ 2 & -1 \end{bmatrix}$.

The inverse of a 2×2 matrix is

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \text{ when } ad - bc \neq 0.$$

Hence, $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -5 & 3 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$.

Conclusion

Inverse matrices are an elegant way of solving linear systems. They do have some drawbacks:

- They are only applicable when the coefficient matrix is square.
- Even in the case of a square matrix, an inverse may not exist.
- They are hard to compute, at least as complicated as doing Gauss-Jordan elimination.

However, they can be useful if

- the coefficient matrix has an obvious inverse,
- you need to solve multiple linear systems with the same coefficients.