Surface Integrals

Math 240 — Calculus III

Summer 2013, Session II

Wednesday, July 3, 2013

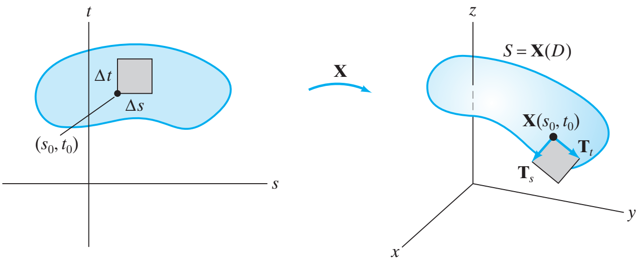
Agenda

1. Scalar surface integrals
   Surface area

2. Vector surface integrals

3. Changing orientation

Scalar surface integrals



Definition

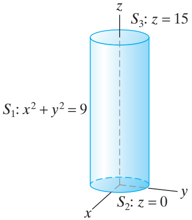
Let $\mathbf{X} : D \subseteq \mathbb{R}^2 \to \mathbb{R}^3$ be a smooth parameterized surface. Let f be a continuous scalar function whose domain includes $S = \mathbf{X}(D)$. The **scalar surface integral** of f along $\mathbf{X}$ is

$$\iint_{\mathbf{X}} f\, dS = \iint_D f(\mathbf{X}(s,t))\, \|\mathbf{T}_s \times \mathbf{T}_t\|\, ds\, dt$$
$$= \iint_D f(\mathbf{X}(s,t))\, \|\mathbf{N}(s,t)\|\, ds\, dt.$$

Scalar surface integrals

Example

Let S be the closed cylinder of radius 3 with axis along the z-axis, top face at $z = 15$, and bottom face at $z = 0$. Let's calculate $\iint_S z\, dS$. Denote the lateral cylindrical face of S by S_1 and the bottom and top faces by S_2 and S_3, respectively.



We compute

$$\iint_{S_1} z\, dS = 675\pi, \quad \iint_{S_2} z\, dS = 0, \text{ and } \iint_{S_3} z\, dS = 135\pi.$$

Therefore,

$$\iint_S z\, dS = \iint_{S_1} z\, dS + \iint_{S_2} z\, dS + \iint_{S_3} z\, dS = 810\pi.$$

Surface area

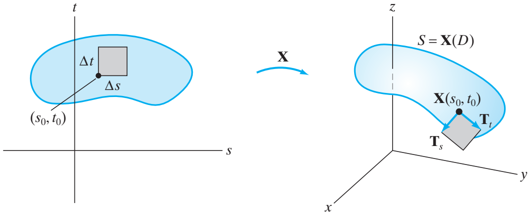


Figure: The quantity $\|\mathbf{T}_s \times \mathbf{T}_t\|$ is the area of the gray square on the right.

Fact

If S is a smooth surface parameterized by $\mathbf{X} : D \subseteq \mathbb{R}^2 \to \mathbb{R}^3$ then the surface area of S is given by

$$\iint_D \|\mathbf{N}\|\, ds\, dt = \iint_D \|\mathbf{T}_s \times \mathbf{T}_t\|\, ds\, dt = \iint_{\mathbf{X}} 1\, dS.$$

Surface area

Example

Recall our parameterization of a sphere:

$$\mathbf{X}(s,t) = r(\cos s)(\sin t)\,\mathbf{i} + r(\sin s)(\sin t)\,\mathbf{j} + r(\cos t)\,\mathbf{k}.$$

We calculate

$$\mathbf{T}_s = -r\sin s\sin t\,\mathbf{i} + r\cos s\sin t\,\mathbf{j},$$
$$\mathbf{T}_t = r\cos s\cos t\,\mathbf{i} + r\sin s\cos t\,\mathbf{j} - r\sin t\,\mathbf{k},$$
$$\mathbf{N} = -r^2\cos s\sin^2 t\,\mathbf{i} - r^2\sin s\sin^2 t\,\mathbf{j} - r^2\sin t\cos t\,\mathbf{k},$$
$$\text{and } \|\mathbf{N}\| = r^2\sin t.$$

Therefore, the surface area of the sphere is

$$\int_0^{\pi}\int_0^{2\pi} r^2\sin t\,ds\,dt = \int_0^{\pi} 2\pi r^2\sin t\,dt = 4\pi r^2.$$

Vector surface integrals

Definition

Let $\mathbf{X} : D \subseteq \mathbb{R}^2 \to \mathbb{R}^3$ be a smooth parameterized surface. Let $\mathbf{F}$ be a continuous vector field whose domain includes $S = \mathbf{X}(D)$. The **vector surface integral** of $\mathbf{F}$ along $\mathbf{X}$ is

$$\iint_{\mathbf{X}} \mathbf{F} \cdot d\mathbf{S} = \iint_D \mathbf{F}(\mathbf{X}(s,t)) \cdot \mathbf{N}(s,t)\, ds\, dt.$$

In physical terms, we can interpret $\mathbf{F}$ as the flow of some kind of fluid. Then the vector surface integral measures the volume of fluid that flows through S per unit time. This is called the **flux** of $\mathbf{F}$ across S.

Changing orientation

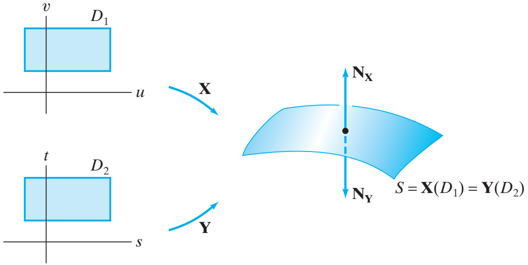


Figure: $\mathbf{X}$ and $\mathbf{Y}$ parameterize the same surface with opposite normal directions.

$$\iint_{\mathbf{Y}} f\,dS = \iint_{\mathbf{X}} f\,dS$$
$$\iint_{\mathbf{Y}} \mathbf{F}\cdot d\mathbf{S} = -\iint_{\mathbf{X}} \mathbf{F}\cdot d\mathbf{S}$$

This can be achieved by exchanging s and t:

$$\mathbf{T}_t \times \mathbf{T}_s = -\left(\mathbf{T}_s \times \mathbf{T}_t\right).$$