Homework assignments

Math 240 — Calculus III

Summer 2013, Session II

Homework is due at the beginning of the first class after it is assigned. You are required to hand in solutions to problems marked with a *. The other homework problems are optional but recommended. I will grade all of the homework that you hand in, but only the starred problems will contribute to your grade. Homework will be accepted late at a penalty of 10% per day (20% for two days, 30% for three days, etc.) but not more than a week late.

I encourage you to work together on the homework problems. However, in order to assign you a grade as an individual, I ask that any work that is handed in for a grade be written up individually. This means that you may work with your fellow students to figure out how to solve a problem, then put away any notes or other written material produced in collaboration and write the solution on your own, using only the understanding you have gained. One way to help you keep work separate is to use colored paper when working with others and make sure that any colored paper is out of sight when you are writing up work that will be handed in.

VC refers to our *Vector Calculus* textbook, DELA to *Differential Equations and Linear Algebra*.

7/1 — These problems are due on 7/2.

1. Calculate the divergence of the following vector fields.

   (a) $\mathbf{F} = x^2\,\mathbf{i} + y^2\,\mathbf{j}$
   Answer: $2x + 2y$

   (b) $\mathbf{F} = (x+y)\,\mathbf{i} + (y+z)\,\mathbf{j} + (x+z)\,\mathbf{k}$
   Answer: 3

   (c) $\mathbf{F} = z\cos\left(e^{y^2}\right)\mathbf{i} + x\sqrt{z^2+1}\,\mathbf{j} + e^{2y}\sin 3x\,\mathbf{k}$
   Answer: 0

2. Find the curl of the following vector fields.

   (a) $\mathbf{F} = (x+yz)\,\mathbf{i} + (y+xz)\,\mathbf{j} + (z+xy)\,\mathbf{k}$
   Answer: $\mathbf{0}$

   $*$ (b) $\mathbf{F} = (e^x + \cos y)\,\mathbf{i} + \left(z^3 + \cos z\right)\mathbf{j} - (x + y\sin z)\,\mathbf{k}$
   Answer: $-3z^2\,\mathbf{i} + \mathbf{j} + (\sin y)\,\mathbf{k}$

   (c) $\mathbf{F} = (\cos yz - x)\,\mathbf{i} + (\cos xz - y)\,\mathbf{j} + (\cos xz - z)\,\mathbf{k}$
   Answer: $x\sin(xz)\mathbf{i} + \big(-y\sin(yz) + z\sin(xz)\big)\mathbf{j} + \big(-z\sin(xz) + z\sin(yz)\big)\mathbf{k}$

3. Calculate $\int_{\mathbf{x}} f\,ds$, where f and $\mathbf{x}$ are as indicated.

   $*$ (a) $f(x,y,z) = 3x + xy + \frac{1}{2}z,\ \mathbf{x}(t) = (\cos 4t, \sin 4t, 3t)\,,\ 0 \le t \le 2\pi$
   Answer: $15\pi^2$

(b) $f(x,y,z) = 4xy + z,\ \mathbf{x}(t) = \left(\frac{1}{3}t^3, t^2, 2t\right),\ 0 \le t \le 1$
Answer: $\frac{28}{9}$

4. Calculate $\int_{\mathbf{x}} \mathbf{F} \cdot d\mathbf{s}$, where the vector field $\mathbf{F}$ and the path $\mathbf{x}$ are given.

   (a) $\mathbf{F} = (y+2)\,\mathbf{i} + x\,\mathbf{j},\ \mathbf{x}(t) = (\sin t, -\cos t),\ 0 \le t \le \frac{\pi}{2}$
   Answer: 2

   (b) $\mathbf{F} = (y-x)\,\mathbf{i} + x^4y^3\,\mathbf{j},\ \mathbf{x}(t) = (t^2, t^3),\ \ -1 \le t \le 1$
   Answer: $\frac{4}{5}$

5. Evaluate $\int_C \left(ye^{xy^2} + \frac{1}{\sqrt{2-x^2}}\right) ds$ where C is the square in the xy-plane with vertices $(1,1)$, $(-1,1)$, $(-1,-1)$, and $(1,-1)$.

   Answer: $\pi + 4$

6. Find the work done by the force field $\mathbf{F} = x^2y\,\mathbf{i} + z\,\mathbf{j} + (2x-y)\,\mathbf{k}$ on a particle as the particle moves along a straight line from $(1,1,1)$ to $(2,-3,3)$.

   Answer: $-\frac{10}{3}$

7. Determine whether the given vector field $\mathbf{F}$ is conservative and, if so, find a scalar potential.

   (a) $\mathbf{F} = z^2\,\mathbf{i} + 2y\,\mathbf{j} + xz\,\mathbf{k}$
   Answer: Not conservative. $\nabla \times \mathbf{F} = z\,\mathbf{j}$.

   (b) $\mathbf{F} = e^x \sin y\,\mathbf{i} + e^x \cos y\,\mathbf{j} + (3z^2+2)\,\mathbf{k}$
   Answer: Conservative. Scalar potential is $f(x,y,z) = e^x \sin y + z^3 + 2z$.

∗ 8. Find the work done by the force field $\mathbf{F} = 2xy\cos z\,\mathbf{i} + x^2\cos z\,\mathbf{j} - x^2y\sin z\,\mathbf{k}$ in moving a particle from $(1,1,\frac{\pi}{2})$ to $(2,3,0)$.

   Answer: 12

7/2 — These problems are due on 7/3.

∗ 1. VC 6.2.4

   Answer: $-\frac{a^2\pi}{2}$

2. Use Green's theorem to evaluate the given line integral.

   (a) $\oint_C e^{2x}\sin 2y\,dx + e^{2x}\cos 2y\,dy$ where C is the ellipse $9(x-1)^2 + 4(y-3)^2 = 36$, oriented counter-clockwise.
   Answer: 0

   (b) $\oint_C \frac{1}{3}y^3\,dx + (xy + xy^2)\,dy$, where C is the boundary of the region in the first quadrant determined by the graphs of $y=0$, $x=y^2$, and $x = 1-y^2$, oriented counter-clockwise.
   Answer: $\frac{1}{8}$

3. Use Green's theorem to compute the area of the pentagon with vertices $(0,4)$, $(4,1)$, $(3,0)$, $(-1,-1)$, and $(-2,2)$.

   Answer: 17

∗ 4. VC 7.1.1

   Answer: (a) $(-1,-4,2)$, (b) $x + 4y - 2z = 5$

5. VC 7.1.4

   Answer: (a) $(1, 0, 2)$, (b) $x + 2z = -1$, (c) $x^2 + y^2 - z^4 = 0$

∗ 6. VC 7.1.15

   Answer: $z = \pm\sqrt{x^2 + y^2 + 1}$

7/3 — These problems are due on 7/8.

1. VC 7.1.25

   Answer: $4\pi a\sqrt{a^2 - b^2}$

∗ 2. VC 7.1.27

   Answer: $\frac{\pi}{24}(65\sqrt{65} - 17\sqrt{17})$

3. VC 7.2.2

   Answer: (a) $\frac{\pi}{3}(6\sqrt{6} - 8)$, (b) $\frac{1}{4}$

4. VC 7.2.13

   Answer: $108\pi + 2\left(\frac{81}{4}\pi\right) = \frac{297}{2}\pi$

∗ 5. VC 7.2.15

   Answer: 36π

6. VC 7.2.28

   Answer: -256π

7/8 — These problems are due on 7/9.

1. VC 7.3.4

   Answer: 4π

   Bonus: Can you simplify the calculation even more by applying Stokes' theorem twice?

   Answer: Integrate $\nabla \times \mathbf{F}$ over the disk $x^2 + y^2 \leq 4$ in the xy-plane.

∗ 2. VC 7.3.7

   Answer: 0

3. VC 7.3.11 — The "rightward-pointing normal" referred to in the problem is the normal with positive **j**-component.

   Answer: 45π

∗ 4. VC 7.3.13

   Answer: (a) Use the identity $\sin 2t = 2\cos t \sin t$. (b) $-\frac{3}{4}\pi$

5. VC 7.3.17

   Answer: $\frac{625}{8}\pi$

$*$ 6. Let S be defined by $z = e^{1-x^2-y^2}$ and $z \geq 1$, oriented by upward-pointing normal, and let $\mathbf{F} = x\,\mathbf{i} + y\,\mathbf{j} + (5-2z)\,\mathbf{k}$.

   (a) Use Gauss' theorem to calculate

   $$\iint_S \mathbf{F} \cdot d\mathbf{S}.$$

   Answer: 3π

   (b) The vector field $\mathbf{F}$, given above, is equal to the curl of $\mathbf{G} = 2yz\,\mathbf{i} + 5x\,\mathbf{j} + xy\,\mathbf{k}$. Use this information and Stokes' theorem to calculate

   $$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S \nabla \times \mathbf{G} \cdot d\mathbf{S}$$

   via a line integral.

7/9 — No homework is due on 7/10 because of the midterm.

The following exercises in *Vector Calculus* refer to problems from the "Miscellaneous Exercises" sections at the end of each chapter.

1. Let $\mathbf{F} = x(y^2+1)\,\mathbf{i} + (ye^x - e^z)\,\mathbf{j} + x^2e^z\,\mathbf{k}$. Compute $\nabla \times \mathbf{F}$ and $\nabla \cdot \mathbf{F}$.

   Answer: $\nabla \times \mathbf{F} = e^z\,\mathbf{i} - 2xe^z\,\mathbf{j} + (ye^x - 2xy)\,\mathbf{k},\ \nabla \cdot \mathbf{F} = y^2 + 1 + e^x + x^2e^z$

2. VC 6.2

   Answer: $2 + 6\pi^2$

3. VC 6.21

   Answer: $\frac{6}{5} - \cos 1 - \sin 1$

4. Consider the vector field $\mathbf{F} = 3x^2y\,\mathbf{i} + x^3\,\mathbf{j} + \cos z\,\mathbf{k}$.

   (a) Compute $\nabla \times \mathbf{F}$.

   Answer: 0

   (b) Find $\int_{\mathbf{x}} \mathbf{F} \cdot d\mathbf{s}$ where $\mathbf{x}(t) = (t^2 - \frac{1}{2}t - 1)\,\mathbf{i} + (t^3 - t - 1)\,\mathbf{j} + \sin(\pi t)\,\mathbf{k}$ and t varies from 0 to 2.

   Answer: 39

5. VC 6.22

   Answer: $-\frac{1}{12}$

6. Find the area of the region enclosed by the curve parameterized by $\mathbf{x}(t) = \sin 2t\,\mathbf{i} + \sin t\,\mathbf{j}$.

   Answer: $\frac{4}{3}$

7. Give a parameterization for the surface obtained by revolving the graph of $y = e^x$ around the x-axis. Find a normal vector and tangent plane to this surface at the point $\left(1, \frac{1}{2}e\sqrt{2}, -\frac{1}{2}e\sqrt{2}\right)$.

   Answer: $\mathbf{X}(s,t) = (s, e^s\cos t, e^s\sin t)\,,\ \mathbf{N} = \left(e^2, -\frac{1}{2}e\sqrt{2}, \frac{1}{2}e\sqrt{2}\right)\,,\ e^2x - \frac{1}{2}e\sqrt{2}y + \frac{1}{2}e\sqrt{2}z = 0$

8. VC 7.11

   Answer: $\frac{11}{10}$

9. Calculate the flux of the vector field $\mathbf{F} = x\,\mathbf{i} - \mathbf{j} + z\,\mathbf{k}$ across the surface with downward-pointing normal given by the equation $z = x\cos y$ with $0 \le x \le 1$ and $\frac{\pi}{4} \le y \le \frac{\pi}{3}$.

   Answer: $\frac{\sqrt{2}-1}{4}$

10. Evaluate the integral
$$\oint_C x^2 e^{5z}\,dx + x\cos y\,dy + 3y\,dz,$$
where C is the curve parameterized by $\mathbf{x}(t) = (2 + 2\cos t)\,\mathbf{j} + (2 + 2\sin t)\,\mathbf{k}$.

    Answer: 12π

11. Evaluate
$$\iint_S \left(y^2 z\,\mathbf{i} + y^3\,\mathbf{j} + xz\,\mathbf{k}\right) \cdot d\mathbf{S},$$
where S is the surface of the cube $-1 \le x \le 1$, $-1 \le y \le 1$, and $0 \le z \le 2$.

    Answer: 8

12. Use Gauss' theorem to compute the volume of the solid obtained by revolving the curve
$$x = \cos t, \quad y = \sin 2t, \quad -\frac{\pi}{2} \le t \le \frac{\pi}{2}$$
about the y-axis.

    Answer: $\frac{\pi^2}{2}$

7/10 — These problems are due on 7/11.

1. DELA 2.1.5
* 2. DELA 2.1.14
3. DELA 2.1.22
4. DELA 2.2.2
* 5. DELA 2.2.5
* 6. DELA 2.2.13
* 7. DELA 2.2.42

7/11 — These problems are due on 7/15.

* 1. DELA 2.3.3
2. DELA 2.3.7
* 3. DELA 2.3.12
4. DELA 2.3.16
* 5. DELA 2.4.9 — Ignore the instruction to "determine the rank of each matrix" in this and the following problems.
6. DELA 2.4.12
* 7. DELA 2.4.21

7/15 — These problems are due on 7/16.

1. DELA 2.5.2
* 2. DELA 2.5.3
* 3. DELA 2.5.13
4. DELA 2.5.16
5. DELA 2.5.23

   For each part, write the rank of the resulting coefficient and augmented matrices.
* 6. DELA 2.6.5
* 7. DELA 2.6.10
8. DELA 2.6.11
9. DELA 2.6.19
10. DELA 2.6.31

7/16 — These problems are due on 7/17.

In problems 1–4, below, in addition to the instructions in the book, indicate whether the matrix is invertible and, if so, find its inverse.

1. DELA 3.1.9
* 2. DELA 3.1.10
3. DELA 3.1.17
* 4. DELA 3.1.18
* 5. DELA 3.1.22
6. DELA 3.2.21
7. DELA 3.2.31
* 8. Let A and B be 4×4 matrices such that $\det(A) = 5$ and $\det(B) = 3$. Compute the determinant of the given matrix.

   (a) A^2B^5

   (b) $(A^{-1}B^2)^3$

   (c) $(\frac{1}{3}A)(2B^T)$

7/17 — These problems are due on 7/18.

* 1. DELA 4.2.1
2. DELA 4.2.2
* 3. DELA 4.2.3
4. DELA 4.2.4
* 5. DELA 4.2.11
6. DELA 4.2.15
7. DELA 4.3.4 — Explain your reasoning for whether or not it is a subspace in questions 7–9.
* 8. DELA 4.3.7
9. DELA 4.3.16
* 10. Determine the span of $\mathbf{v} = (-3, -1)$ in $\mathbb{R}^2$ and sketch it.
11. DELA 4.4.20

7/18 — These problems are due on 7/22.

1. DELA 4.4.12
2. DELA 4.4.14
* 3. DELA 4.4.22 — If the given vector lies in $\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2\}$, express it as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$.
4. DELA 4.4.24 — If the given vector lies in $\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2\}$, express it as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$.
5. DELA 4.5.10
6. DELA 4.5.16 — If the set is not linearly independent, give a linear dependence relation.
* 7. DELA 4.5.17 — If the set is not linearly independent, give a linear dependence relation.
8. DELA 4.5.29
9. DELA 4.5.32
10. DELA 4.6.2
11. DELA 4.6.9 — Provide a basis for the null space of A.
* 12. DELA 4.6.11 — Provide a basis for the null space of A.

7/22 — These problems are due on 7/23.

∗ 1. DELA 4.8.3
2. DELA 4.8.7
∗ 3. DELA 4.8.10
4. DELA 4.9.3
5. DELA 4.9.4
∗ 6. DELA 4.9.10 — Also determine $\mathrm{rank}(A)$ and $\mathrm{nullity}(A)$. (Hint: you only need to compute one.)
7. DELA 4.9.11 — Also determine $\mathrm{rank}(A)$ and $\mathrm{nullity}(A)$.

7/23 — These problems are due on 7/24.

∗ 1. DELA 5.1.5
2. DELA 5.1.6
∗ 3. DELA 5.1.12 — Also determine the domain and codomain of the linear transformation.
4. DELA 5.1.24
5. DELA 5.2.1
6. DELA 5.2.2
∗ 7. DELA 5.3.4
8. DELA 5.3.7
9. DELA 5.5.3
10. DELA 5.5.5
∗ 11. DELA 5.5.20

7/24 — These problems are due on 7/25.

∗ 1. DELA 5.6.8
2. DELA 5.6.9
∗ 3. DELA 5.6.13
∗ 4. DELA 5.6.16
5. DELA 5.6.17
∗ 6. DELA 5.8.16
7. DELA 5.8.17

7/25 — No homework is due on 7/29 because of the midterm.

1. Determine whether the given set is a vector space. If not, give at least one axiom that is not satisfied. Unless otherwise stated, assume that vector addition and scalar multiplication are the ordinary operations defined on the set.

   (a) The set of vectors $\{(a, b) \in \mathbb{R}^2 : b = 3a + 1\}$
   Answer: This is not a vector space. It does not contain the zero vector, and is not closed under either addition or scalar multiplication.

   (b) The set of vectors $\{(a, b) \in \mathbb{R}^2\}$ with scalar multiplication defined by $k(a, b) = (ka, b)$
   Answer: This is not a vector space. The scalar multiplication defined above does not distribute over the usual addition of vectors.

   $$(r + s)(a, b) = ((r + s)a, b) = (ra + sa, b)$$
   $$\text{but } r(a, b) + s(a, b) = (ra, b) + (sa, b) = (ra + sa, 2b)$$

   (c) The set of vectors $\{(a, b) \in \mathbb{R}^2\}$ with scalar multiplication defined by $k(a, b) = (ka, 0)$
   Answer: This is not a vector space. It does not obey the identity property of scalar multiplication because
   $$1(a, b) = (1a, 0) = (a, 0) \neq (a, b).$$

   (d) The set of real numbers, with addition defined by $\mathbf{x} + \mathbf{y} = x - y$
   Answer: This is not a vector space. This method of vector addition is neither associative nor commutative.

   (e) The set of complex numbers $a + bi$, where $i^2 = -1$, with addition and scalar multiplicatin defined by

   $$(a_1 + b_1 i) + (a_2 + b_2 i) = (a_1 + a_2) + (b_1 + b_2)i$$
   $$k(a + bi) = ka + kbi, \text{ for any real number } k$$

   Answer: This is a vector space. It is the real vector space of complex numbers.

2. Determine whether or not the given set is a subspace of the indicated vector space.

   (a) $\{\mathbf{x} \in \mathbb{R}^3 : \|\mathbf{x}\| = 1\}$
   Answer: This is not a subspace of $\mathbb{R}^3$. It does not contain the zero vector $\mathbf{0} = (0, 0, 0)$ and it is not closed under either addition or scalar multiplication.

   (b) All polynomials in P_2 that are divisible by $x - 2$
   Answer: This is a subspace of P_2.

   (c) $\{f \in C^0[a, b] : \int_a^b f(x)\,dx = 0\}$
   Remember that $C^0[a, b]$ is the vector space of continuous, real-valued functions defined on the closed interval $[a, b]$ with $a < b$.
   Answer: This is a subspace of $C^0[a, b]$. It is the kernel of the linear transformation $T : C^0[a, b] \to \mathbb{R}$ defined by $T(f) = \int_a^b f(x)\,dx$.

3. If $A = \begin{bmatrix} 1 & 4 \\ 5 & 10 \\ 8 & 12 \end{bmatrix}$ and $B = \begin{bmatrix} -4 & 6 & -3 \\ 1 & -3 & 2 \end{bmatrix}$, determine (a) AB and (b) BA.

   Answer: (a) $AB = \begin{bmatrix} 0 & -6 & 5 \\ -10 & 0 & 5 \\ -20 & 12 & 0 \end{bmatrix}$, (b) $BA = \begin{bmatrix} 2 & 8 \\ 2 & -2 \end{bmatrix}$

4. Use either Gaussian elimination or Gauss-Jordan elimination to solve the given system or show that no solution exists.

(a)

$$\begin{aligned} x_1 - x_2 - x_3 &= -3 \\ 2x_1 + 3x_2 + 5x_3 &= 7 \\ x_1 - 2x_2 + 3x_3 &= -11 \end{aligned}$$

Answer: $x_1 = 0,\ x_2 = 4,\ x_3 = -1$

(b)

$$\begin{aligned} x_1 + x_2 + x_3 &= 0 \\ x_1 + x_2 + 3x_3 &= 0 \end{aligned}$$

Answer: $x_1 = t,\ x_2 = -t,\ x_3 = 0$ for any $t \in \mathbb{R}$

(c)

$$\begin{aligned} x_1 - x_2 - x_3 &= 8 \\ x_1 - x_2 + x_3 &= 3 \\ -x_1 + x_2 + x_3 &= 4 \end{aligned}$$

Answer: The system is inconsistent; there is no solution.

(d)

$$\begin{aligned} x_1 + 2x_2 \qquad + x_4 &= 0 \\ 4x_1 + 9x_2 + x_3 + 12x_4 &= 0 \\ 3x_1 + 9x_2 + 6x_3 + 21x_4 &= 0 \\ x_1 + 3x_2 + x_3 + 9x_4 &= 0 \end{aligned}$$

Answer: $x_1 = 19t,\ x_2 = -10t,\ x_3 = 2t,\ x_4 = t$ for any $t \in \mathbb{R}$

5. Determine the rank of the given matrix.

(a) $\begin{bmatrix} 3 & -1 \\ 1 & 3 \end{bmatrix}$

Answer: 2

(b) $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 4 \\ 1 & 4 & 1 \end{bmatrix}$

Answer: 3

(c) $\begin{bmatrix} 0 & 2 & 4 & 2 & 2 \\ 4 & 1 & 0 & 5 & 1 \\ 2 & 1 & \frac{2}{3} & 3 & \frac{1}{3} \\ 6 & 6 & 6 & 12 & 0 \end{bmatrix}$

Answer: 3

6. Determine whether the given set of vectors is linearly dependent or linearly independent.

(a) $\mathbf{v}_1 = (1, 2, 3)$, $\mathbf{v}_2 = (1, 0, 1)$, $\mathbf{v}_3 = (1, -1, 5)$
Answer: These vectors are linearly independent.

(b) $\mathbf{v}_1 = (2, 6, 3)$, $\mathbf{v}_2 = (1, -1, 4)$, $\mathbf{v}_3 = (3, 2, 1)$, $\mathbf{v}_4 = (2, 5, 4)$
Answer: These vectors are linearly dependent. They are vectors in $\mathbb{R}^3$, which is a 3-dimensional vector space. Any set of more than 3 vectors in $\mathbb{R}^3$ is linearly dependent.

(c) $\mathbf{v}_1 = (1, -1, 3, -1)$, $\mathbf{v}_2 = (1, -1, 4, 2)$, $\mathbf{v}_3 = (1, -1, 5, 7)$.
Answer: These vectors are linearly independent.

(d) $\mathbf{v}_1 = (2, 1, 1, 5)$, $\mathbf{v}_2 = (2, 2, 1, 1)$, $\mathbf{v}_3 = (3, -1, 6, 1)$, $\mathbf{v}_4 = (1, 1, 1, -1)$
Answer: These vectors are linearly independent.

7. Determine a basis for the subspace of $\mathbb{R}^n$ spanned by the given set of vectors.

(a) $\{(1, 3, 3),\ (1, 5, -1),\ (2, 7, 4),\ (1, 4, 1)\}$
Answer: $\{(1, 3, 3),\ (1, 5, -1)\}$ (answers may vary)

(b) $\{(1, 1, -1, 2),\ (2, 1, 3, -4),\ (1, 2, -6, 10)\}$
Answer: $\{(1, 1, -1, 2),\ (2, 1, 3, -4)\}$ (answers may vary)

8. Evaluate the determinant of the given matrix.

(a) $\begin{bmatrix} 0 & 2 & 0 \\ 3 & 0 & 1 \\ 0 & 5 & 8 \end{bmatrix}$

Answer: -48

(b) $\begin{bmatrix} 3 & 0 & 2 \\ 2 & 7 & 1 \\ 2 & 6 & 4 \end{bmatrix}$

Answer: 62

(c) $\begin{bmatrix} 4 & 5 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}$

Answer: 0

(d) $\begin{bmatrix} -2 & -1 & 4 \\ -3 & 6 & 1 \\ -3 & 4 & 8 \end{bmatrix}$

Answer: -85

(e) $\begin{bmatrix} 6 & 1 & 8 & 10 \\ 0 & \frac{2}{3} & 7 & 2 \\ 0 & 0 & -4 & 9 \\ 0 & 0 & 0 & -5 \end{bmatrix}$

Answer: 80

(f) $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 5 & 7 \\ 2 & 3 & 6 & 7 \\ 1 & 5 & 8 & 20 \end{bmatrix}$

Answer: 16

9. Find the values of λ that satisfy the equation

$$\begin{vmatrix} -3-\lambda & 10 \\ 2 & 5-\lambda \end{vmatrix} = 0.$$

Answer: $\lambda = -5, 7$

10. If $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 5$, evaluate the determinant of the matrix

$$\begin{bmatrix} 2a_1 & a_2 & a_3 \\ 6b_1 & 3b_2 & 3b_3 \\ 2c_1 & c_2 & c_3 \end{bmatrix}.$$

Answer: 30

11. Recall that a square matrix A is said to be **skew-symmetric** if $A^T = -A$. If A is a 5×5 skew-symmetric matrix, show that $\det(A) = 0$.

Answer: We know that $\det\left(A^T\right) = \det(A)$. Also, if A is 5×5, then $\det(-A) = (-1)^5 \det(A)$. Putting these together with the information $A^T = -A$, we get

$$\det(A) = \det\left(A^T\right) = \det(-A) = (-1)^5 \det(A) = -\det(A).$$

The only number that is equal to its negative is 0.

12. Determine whether the given matrix is singular or nonsingular. If it is nonsingular, find its inverse.

(a) $\begin{bmatrix} 6 & 0 \\ -3 & 2 \end{bmatrix}$

Answer: $\begin{bmatrix} 1/6 & 0 \\ 1/4 & 1/2 \end{bmatrix}$

(b) $\begin{bmatrix} -2\pi & -\pi \\ -\pi & \pi \end{bmatrix}$

Answer: $\dfrac{-1}{3\pi}\begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 2 & 3 \\ 0 & -4 & 2 \\ -1 & 5 & 1 \end{bmatrix}$

Answer:
$\begin{bmatrix} 7/15 & -13/30 & -8/15 \\ 1/15 & -2/15 & 1/15 \\ 2/15 & 7/30 & 2/15 \end{bmatrix}$

(d) $\begin{bmatrix} 3 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & -2 \end{bmatrix}$

Answer:
$\begin{bmatrix} 1/3 & 0 & 0 \\ 0 & 1/6 & 0 \\ 0 & 0 & -1/2 \end{bmatrix}$

(e) $\begin{bmatrix} 4 & 2 & 3 \\ 2 & 1 & 0 \\ -1 & -2 & 0 \end{bmatrix}$

Answer:
$\dfrac{1}{3}\begin{bmatrix} 0 & 2 & 1 \\ 0 & -1 & -2 \\ 1 & -2 & 0 \end{bmatrix}$

(f) $\begin{bmatrix} -1 & -1 & 1 \\ -1 & 5 & 0 \\ 0 & 6 & -1 \end{bmatrix}$

Answer: This matrix is singular.

13. Use an inverse matrix to solve the linear system

$$\begin{aligned} x_1 + 2x_2 + 2x_3 &= 1 \\ x_1 - 2x_2 + 2x_3 &= -3 \\ 3x_1 - x_2 + 5x_3 &= 7 \end{aligned}$$

Answer: $x_1 = 21,\ x_2 = 1,\ x_3 = -11$

14. Write the linear system

$$\begin{aligned} 7x_1 - 2x_2 &= b_1, \\ 3x_1 - 2x_2 &= b_2, \end{aligned}$$

in the form $A\mathbf{x} = \mathbf{b}$. Use $\mathbf{x} = A^{-1}\mathbf{b}$ to solve the system for each $\mathbf{b}$:

$$\mathbf{b} = \begin{bmatrix} 5 \\ 4 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 10 \\ 50 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 0 \\ -20 \end{bmatrix}.$$

Answer: $A\mathbf{x} = \mathbf{b}$ where $A = \begin{bmatrix} 7 & -2 \\ 3 & -2 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$.

$$A^{-1} = \begin{bmatrix} 1/4 & -1/4 \\ 3/8 & -7/8 \end{bmatrix}, \quad A^{-1}\begin{bmatrix} 5 \\ 4 \end{bmatrix} = \begin{bmatrix} 1/4 \\ -13/8 \end{bmatrix}, \quad A^{-1}\begin{bmatrix} 10 \\ 50 \end{bmatrix} = \begin{bmatrix} -10 \\ -40 \end{bmatrix}, \quad A^{-1}\begin{bmatrix} 0 \\ -20 \end{bmatrix} = \begin{bmatrix} 5 \\ 35/2 \end{bmatrix}$$

15. Determine the matrix representation for the given linear transformation T relative to the ordered bases B and C.

(a) $T : M_2(\mathbb{R}) \to P_3$ given by $T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = (a-d) + 3bx^2 + (c-a)x^3$ with

i. $B = \left\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right\}$ and $C = \{1, x, x^2, x^3\}$,

Answer: $\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ -1 & 0 & 1 & 0 \end{bmatrix}$

ii. $B = \left\{\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right\}$ and $C = \{x, 1, x^3, x^2\}$.

Answer: $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$

(b) $T : V \to V$, where $V = \mathrm{span}\{e^{2x}, e^{-3x}\}$, given by $T(f) = f'$ with

i. $B = C = \{e^{2x}, e^{-3x}\}$,

Answer: $\begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix}$

ii. $B = \{e^{2x} - 3e^{-3x}, 2e^{-3x}\}$ and $C = \{e^{2x} + e^{-3x}, -e^{2x}\}$.

Answer: $\begin{bmatrix} 9 & -6 \\ 7 & -6 \end{bmatrix}$

16. Determine which of the indicated column vectors are eigenvectors of the given matrix A. Give the corresponding eigenvalue for each one that is.

(a) $A = \begin{bmatrix} 4 & 2 \\ 5 & 1 \end{bmatrix}$, $\mathbf{v}_1 = (5, -2),\ \mathbf{v}_2 = (2, 5),\ \mathbf{v}_3 = (-2, 5)$

Answer: $\mathbf{v}_3$ is an eigenvector for the eigenvalue $\lambda = -1$.

(b) $A = \begin{bmatrix} 2 & -1 \\ 2 & -2 \end{bmatrix}$, $\mathbf{v}_1 = (1, 2-\sqrt{2}),\ \mathbf{v}_2 = (2+\sqrt{2}, 2),\ \mathbf{v}_3 = (\sqrt{2}, -\sqrt{2})$

Answer: $\mathbf{v}_1$ and $\mathbf{v}_2$ are eigenvectors for the eigenvalue $\lambda = \sqrt{2}$.

(c) $A = \begin{bmatrix} 2 & 8 \\ -1 & -2 \end{bmatrix}$, $\mathbf{v}_1 = (0, 0),\ \mathbf{v}_2 = (2+2i, -1),\ \mathbf{v}_3 = (2+2i, 1)$

Answer: $\mathbf{v}_2$ is an eigenvector for the eigenvalue $\lambda = 2i$.

Note: The zero vector is not allowed as an eigenvector.

(d) $A = \begin{bmatrix} -1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 3 & -1 \end{bmatrix}$, $\mathbf{v}_1 = (-1, 4, 3),\ \mathbf{v}_2 = (1, 4, 3),\ \mathbf{v}_3 = (3, 1, 4)$

Answer: $\mathbf{v}_2$ is an eigenvector for the eigenvalue $\lambda = 3$.

17. Find the eigenvalues and eigenvectors of the given matrix.

(a) $\begin{bmatrix} -1 & 2 \\ -7 & 8 \end{bmatrix}$

Answer: $\lambda_1 = 1,\ \mathbf{v}_1 = (1, 1),\ \lambda_2 = 6,\ \mathbf{v}_2 = (2, 7)$

(b) $\begin{bmatrix} -1 & 2 \\ -5 & 1 \end{bmatrix}$

Answer: $\lambda_1 = 3i,\ \mathbf{v}_1 = (2, 1+3i),\ \lambda_2 = -3i,\ \mathbf{v}_2 = (2, 1-3i)$

(c) $\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

Answer: $\lambda_1 = 1+i,\ \mathbf{v}_1 = (i, 1),\ \lambda_2 = 1-i,\ \mathbf{v}_2 = (-i, 1)$

(d) $\begin{bmatrix} 5 & -1 & 0 \\ 0 & -5 & 9 \\ 5 & -1 & 0 \end{bmatrix}$

Answer: $\lambda_1 = 4,\ \mathbf{v}_1 = (1, 1, 1),\ \lambda_2 = -4,\ \mathbf{v}_2 = (1, 9, 1),\ \lambda_3 = 0,\ \mathbf{v}_3 = (9, 45, 25)$

(e) $\begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 1 & 1 & -1 \end{bmatrix}$

Answer: $\lambda_1 = -1,\ \mathbf{v}_1 = (1, -1, 1),\ \lambda_2 = i,\ \mathbf{v}_2 = (i, 1, 1),\ \lambda_3 = -i,\ \mathbf{v}_3 = (-i, 1, 1)$

(f) $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 6 \\ 0 & 0 & -7 \end{bmatrix}$

Answer: $\lambda_1 = 1,\ \mathbf{v}_1 = (1, 0, 0),\ \lambda_2 = 5,\ \mathbf{v}_2 = (1, 2, 0),\ \lambda_3 = -7,\ \mathbf{v}_3 = (1, 2, -4)$

18. Determine whether the given matrix A is diagonalizable. If so, find the matrix P that diagonalizes A and the diagonal matrix D such that $D = P^{-1}AP$.

(a) $\begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix}$

Answer: This matrix has one eigenvalue with algebraic multiplicity 2, but only one linearly independent eigenvector. It is defective and therefore not diagonalizable.

(b) $\begin{bmatrix} -9 & 13 \\ -2 & 6 \end{bmatrix}$

Answer:

$$P = \begin{bmatrix} 1 & 13 \\ 1 & 2 \end{bmatrix}, \ D = \begin{bmatrix} 4 & 0 \\ 0 & -7 \end{bmatrix}$$

(c) $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

Answer:

$$P = \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix}, \ D = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$$

(d) $\begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix}$

Answer:

$$P = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix}, \ D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

(e) $\begin{bmatrix} 1 & 3 & -1 \\ 0 & 2 & 4 \\ 0 & 0 & 1 \end{bmatrix}$

Answer: This matrix has two eigenvalues with algebraic multiplicities of 1 and 2, respectively. The latter has only one linearly independent eigenvector, hence the matrix is defective and not diagonalizable.

(f) $\begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Answer:

$$P = \begin{bmatrix} 1+\sqrt{5} & 1-\sqrt{5} & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \ D = \begin{bmatrix} \sqrt{5} & 0 & 0 \\ 0 & -\sqrt{5} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

19. Use diagonalization to solve the given system of differential equations.

(a) $\mathbf{x}' = \begin{bmatrix} 5 & 6 \\ 3 & -2 \end{bmatrix} \mathbf{x}$

Answer:

$$\mathbf{x} = \begin{bmatrix} 3c^1 e^{7t} + 2c_2 e^{-4t} \\ c_1 e^{7t} - 3c_2 e^{-4t} \end{bmatrix}$$

(b) $\mathbf{x}' = \begin{bmatrix} -1 & 3 & 0 \\ 3 & -1 & 0 \\ -2 & -2 & 6 \end{bmatrix} \mathbf{x}$

Answer:

$$\mathbf{x} = \begin{bmatrix} c_1 e^{2t} + c_2 e^{-4t} \\ c_1 e^{2t} - c_2 e^{-4t} \\ c_1 e^{2t} + c_3 e^{6t} \end{bmatrix}$$

(c) $\mathbf{x}' = \begin{bmatrix} 0 & 2 & 0 \\ 2 & 0 & 2 \\ 0 & 2 & 0 \end{bmatrix} \mathbf{x}$

Answer:

$$\mathbf{x} = \begin{bmatrix} c_1 e^{2\sqrt{2}t} + c_2 e^{-2\sqrt{2}t} + c_3 \\ \sqrt{2} c_1 e^{2\sqrt{2}t} - \sqrt{2} c_2 e^{-2\sqrt{2}t} \\ c_1 e^{2\sqrt{2}t} + c_2 e^{-2\sqrt{2}t} - c_3 \end{bmatrix}$$

7/29 — These problems are due on 7/30.

∗ 1. DELA 7.1.14

2. DELA 7.1.15

3. DELA 7.2.3

∗ 4. DELA 7.2.4

5. DELA 7.2.9

$*$ 6. Verify that

$$\mathbf{x}_1(t) = \begin{bmatrix} -e^t \\ e^t \end{bmatrix} \text{ and } \mathbf{x}_2(t) = \begin{bmatrix} e^{7t} \\ 5e^{7t} \end{bmatrix}$$

are solutions of the system $\mathbf{x}'(t) = A(t)\mathbf{x}(t)$ where

$$A(t) = \begin{bmatrix} 2 & 1 \\ 5 & 6 \end{bmatrix}.$$

Find the general solution to the system (remember to check linear independence) and find the particular solution satisfying the initial condition

$$\mathbf{x}(0) = \begin{bmatrix} -2 \\ 1 \end{bmatrix}.$$

7. DELA 7.3.4

Is there any point $t \in \mathbb{R}$ where $W[\mathbf{x}_1, \mathbf{x}_2](t) = 0$? How can you reconcile this with Theorem 7.3.4?

7/30 — These problems are due on 7/31.

1. DELA 7.4.1
2. DELA 7.4.3

$*$ 3. DELA 7.4.4

4. DELA 7.4.9

$*$ 5. DELA 7.4.10

6. DELA 7.4.11

$*$ 7. Solve the problem at the top of DELA p. 534. Use the following values for the constants:

$$m_1 = 2\,\text{kg}, \quad m_2 = 1\,\text{kg}, \quad k_1 = 4\,\text{N/m}, \quad k_2 = 2\,\text{N/m}.$$

Here are the equations with the given constants.

$$\begin{aligned} 2\frac{d^2x}{dt^2} &= -4x + 2(y-x), \\ \frac{d^2y}{dt^2} &= -2(y-x). \end{aligned}$$

You should (a) find the general solution and (b) find the particular solution satisfying the initial condition

$$x(0) = 0\,\text{m}, \quad y(0) = 2\,\text{m}, \quad x'(0) = 0\,\text{m/s} = y'(0).$$

7/31 — These problems are due on 8/1.

Determine the Jordan canonical form J for the given matrix A, and find the invertible matrix S such that $J = S^{-1}AS$.

1. $A = \begin{bmatrix} 0 & -2 \\ 2 & 4 \end{bmatrix}$

2. $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

∗ 3. $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & -1 \end{bmatrix}$

∗ 4. $A = \begin{bmatrix} 5 & 0 & -1 \\ 1 & 4 & -1 \\ 1 & 0 & 3 \end{bmatrix}$

∗ 5. $A = \begin{bmatrix} 7 & -2 & 2 \\ 0 & 4 & -1 \\ -1 & 1 & 4 \end{bmatrix}$

6. $A = \begin{bmatrix} 2 & -4 & 2 & 2 \\ -2 & 0 & 1 & 3 \\ -2 & -2 & 3 & 3 \\ -2 & -6 & 3 & 7 \end{bmatrix}$ (Hint: the characteristic polynomial is $(2-\lambda)^2(4-\lambda)^2$.)

8/1 — These problems are due on 8/5.

1. DELA 7.5.2
2. DELA 7.5.11
∗ 3. DELA 7.5.12
∗ 4. DELA 7.5.14
5. DELA 6.1.9
∗ 6. DELA 6.1.10
∗ 7. DELA 6.1.16
∗ 8. DELA 6.1.20
9. DELA 6.1.21

8/5 — These problems are due on 8/6.

1. DELA 6.1.31
∗ 2. DELA 6.1.36
3. DELA 6.2.6
4. DELA 6.2.7
5. DELA 6.2.21
∗ 6. DELA 6.2.22
∗ 7. DELA 6.3.4
8. DELA 6.3.7

9. DELA 6.3.17
10. ∗ DELA 6.3.32 (You may use annihilators if you wish.)

8/6 — These problems are due on 8/7.

1. ∗ DELA 6.4.2
2. DELA 6.4.7
3. ∗ DELA 6.4.8
4. DELA 6.4.10