3.1

T/F

2. F

Problems

2. In order for T to be a linear transformation, we require $T(\mathbf{v}+\mathbf{w}) = T(\mathbf{v}) + T(\mathbf{w})$ and $T(c\mathbf{v}) = cT(\mathbf{v})$ for all scalars c and vectors $\mathbf{v}$ and $\mathbf{w}$. If $\mathbf{v} = (x_1, x_2, x_3)$ and $\mathbf{w} = (y_1, y_2, y_3)$ then

$$\begin{aligned} T(\mathbf{v}+\mathbf{w}) &= T(x_1+y_1,\, x_2+y_2,\, x_3+y_3) \\ &= ((x_1+y_1)+3(x_2+y_2)+(x_3+y_3),\, (x_1+y_1)-(x_2+y_2)) \\ &= ((x_1+3x_2+x_3)+(y_1+3y_2+y_3),\, (x_1-x_2)+(y_1-y_2)) \\ &= (x_1+3x_2+x_3,\, x_1-x_2)+(y_1+3y_2+y_3,\, y_1-y_2) \\ &= T(\mathbf{v})+T(\mathbf{w}) \end{aligned}$$

and

$$\begin{aligned} T(c\mathbf{v}) &= (cx_1+3(cx_2)+cx_3,\, cx_1-cx_2) \\ &= (c(x_1+3x_2+x_3),\, c(x_1-x_2)) \\ &= c\,(x_1+3x_2+x_3,\, x_1-x_2) = cT(\mathbf{v}). \end{aligned}$$

10. This mapping T satisfies neither one of our criteria for linear transformations. If A and B are 2×2 matrices then in general

$$T(A+B) = \det(A+B) \neq \det A + \det B = T(A)+T(B).$$

In particular, if A and B are the 2×2 identity matrix then we have a counterexample:

$$\det(A+B) = \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} = 4 \neq 2 = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = \det A + \det B.$$

Additionally, scalars do not pull out like they're supposed to. If T is a linear transformation then for any scalar c we have $T(cA) = cT(A)$. For this T we have instead $T(cA) = c^2 T(A)$. In fact, if A is the identity matrix and $c=2$ then this is the same counterexample as above.

12. The matrix for the linear transformation T is

$$\begin{bmatrix} T(\mathbf{e}_1) & T(\mathbf{e}_2) & \ldots & T(\mathbf{e}_n) \end{bmatrix}.$$

In this case $n = 2$. The standard basis is

$$\begin{aligned}\mathbf{e}_1 &= (1,0)\\ \mathbf{e}_2 &= (0,1)\end{aligned}$$

and when we apply T we get

$$\begin{aligned}T(\mathbf{e}_1) &= (1,2,1)\\ T(\mathbf{e}_2) &= (3,-7,0)\end{aligned}$$

so the matrix of T is

$$\begin{bmatrix}1 & 3\\ 2 & -7\\ 1 & 0\end{bmatrix}.$$

24. To find the matrix for T we need to know what T does to the standard basis $\mathbf{e}_1 = (1,0)$ and $\mathbf{e}_2 = (0,1)$. We can write $\mathbf{e}_1$ as a linear combination of $(-1,1)$ and $(1,2)$:

$$\mathbf{e}_1 = (1,0) = \tfrac{1}{3}(1,2) - \tfrac{2}{3}(-1,1)$$

so

$$\begin{aligned}T(\mathbf{e}_1) = T(1,0) &= T\left(\tfrac{1}{3}(1,2) - \tfrac{2}{3}(-1,1)\right)\\ &= \tfrac{1}{3}T(1,2) - \tfrac{2}{3}T(-1,1)\\ &= \tfrac{1}{3}(-3,1,1,1) - \tfrac{2}{3}(1,0,-2,2)\\ &= \left(-\tfrac{5}{3}, \tfrac{1}{3}, \tfrac{5}{3}, -1\right).\end{aligned}$$

Similarly,

$$\mathbf{e}_2 = (0,1) = \tfrac{1}{3}(1,2) + \tfrac{1}{3}(-1,1)$$

so

$$\begin{aligned}T(\mathbf{e}_2) = T(0,1) &= T\left(\tfrac{1}{3}(1,2) + \tfrac{1}{3}(-1,1)\right)\\ &= \tfrac{1}{3}T(1,2) + \tfrac{1}{3}T(-1,1)\\ &= \tfrac{1}{3}(-3,1,1,1) + \tfrac{1}{3}(1,0,-2,2)\\ &= \left(-\tfrac{2}{3}, \tfrac{1}{3}, -\tfrac{1}{3}, 1\right).\end{aligned}$$

Thus, the matrix for T is

$$\begin{bmatrix}-\frac{5}{3} & -\frac{2}{3}\\ \frac{1}{3} & \frac{1}{3}\\ \frac{5}{3} & -\frac{1}{3}\\ -1 & 1\end{bmatrix}.$$

34. Let $\mathbf{v}, \mathbf{w} \in V$ and d be a scalar.

$$\begin{aligned}(T_1 + T_2)(\mathbf{v} + \mathbf{w}) &= T_1(\mathbf{v} + \mathbf{w}) + T_2(\mathbf{v} + \mathbf{w})\\ &= (T_1(\mathbf{v}) + T_1(\mathbf{w})) + (T_2(\mathbf{v}) + T_2(\mathbf{w}))\\ &= (T_1(\mathbf{v}) + T_2(\mathbf{v})) + (T_2(\mathbf{w}) + T_2(\mathbf{w}))\\ &= (T_1 + T_2)(\mathbf{v}) + (T_1 + T_2)(\mathbf{w})\end{aligned}$$

and

$$\begin{aligned}(T_1 + T_2)(d\mathbf{v}) &= T_1(d\mathbf{v}) + T_2(d\mathbf{v}) \\ &= d\,T_1(\mathbf{v}) + d\,T_2(\mathbf{v}) \\ &= d\,(T_1(\mathbf{v}) + T_2(\mathbf{v})) \\ &= d\,(T_1 + T_2)(\mathbf{v})\end{aligned}$$

show that $(T_1 + T_2)$ is a linear transformation.

$$\begin{aligned}(c\,T_1)(\mathbf{v} + \mathbf{w}) &= c\,T_1(\mathbf{v} + \mathbf{w}) \\ &= c\,(T_1(\mathbf{v}) + T_1(\mathbf{w})) \\ &= c\,T_1(\mathbf{v}) + c\,T_1(\mathbf{w}) \\ &= (c\,T_1)(\mathbf{v}) + (c\,T_1)(\mathbf{w})\end{aligned}$$

and

$$\begin{aligned}(c\,T_1)(d\mathbf{v}) &= c\,T_1(d\mathbf{v}) \\ &= cd\,T_1(\mathbf{v}) \\ &= d\,(c\,T_1(\mathbf{v})) \\ &= d\,(c\,T_1)\,(\mathbf{v})\end{aligned}$$

show that $c\,T_1$ is a linear transformation.

3.1

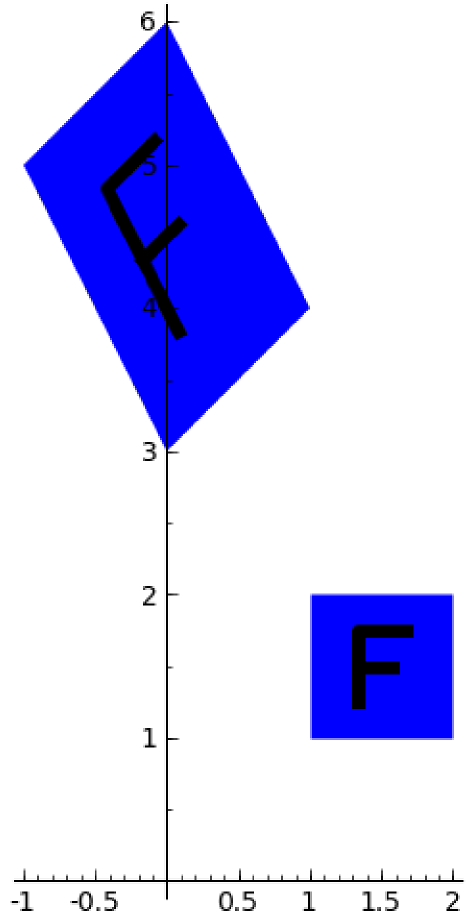
T/F

4. T

6. F

Problems

2. Here's what the linear transformation of A does to the square ("F" added to show orientation):



6. First, we use elementary row operations to reduce A to the identity.

$$\begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix} \xrightarrow{P_{12}} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \xrightarrow{M_1\left(\frac{1}{2}\right)} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \xrightarrow{M_2\left(\frac{1}{2}\right)} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

This tells us that

$$A = P_{12}^{-1} M_1 \left(\tfrac{1}{2}\right)^{-1} M_2 \left(\tfrac{1}{2}\right)^{-1} = P_{12} M_1(2) M_2(2).$$

Now we use the correspondences given in section 5.2 to read off that A is first a stretch in the y-direction, then a stretch in the x-direction, and finally a reflection in the line $y = x$. (Actually, the order of the two stretches does not matter; we say that they *commute* with each other.)

10. Use the same method as in #6.

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \xrightarrow{A_{12}(-3)} \begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix} \xrightarrow{M_2\left(-\frac{1}{2}\right)} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \xrightarrow{A_{21}(-2)} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

which means that

$$\begin{aligned} A &= A_{12}(-3)^{-1} M_2 \left(-\tfrac{1}{2}\right)^{-1} A_{21}(-2)^{-1} \\ &= A_{12}(3) M_2(-2) A_{21}(2) \\ &= A_{12}(3) M_2(2) M_2(-1) A_{21}(2). \end{aligned}$$

So the linear transformation given by A is first a shear parallel to the x-axis, then reflection in the x-axis, then a stretch in the y-direction, and finally a shear parallel to the y-axis.