VC 7.3

4. First, we evaluate $\iint_S \nabla \times \mathbf{F} \cdot d\mathbf{S}$. Parameterize S by

$$\mathbf{X}(r,\theta) = \left(r\cos\theta,\ r\sin\theta,\ -\sqrt{4-r^2}\right)$$

where r goes from 0 to 2 and θ goes from 0 to 2π. The tangent vectors are

$$\mathbf{T}_r = \left(\cos\theta,\ \sin\theta,\ \frac{r}{\sqrt{4-r^2}}\right)$$
$$\mathbf{T}_\theta = (-r\sin\theta,\ r\cos\theta,\ 0)$$

and the normal vector is

$$\mathbf{T}_\theta \times \mathbf{T}_r = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -r\sin\theta & r\cos\theta & 0 \\ \cos\theta & \sin\theta & \frac{r}{\sqrt{4-r^2}} \end{vmatrix} = \frac{r^2}{\sqrt{4-r^2}}(\cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}) - r\mathbf{k}.$$

Notice that, since $r \geq 0$, the $\mathbf{k}$ component of this normal vector is ≤ 0; we have the downward pointing normal.

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2y - z & x + y^2 - z & 4y - 3x \end{vmatrix} = 5\mathbf{i} + 2\mathbf{j} - \mathbf{k}$$

Now we can integrate:

$$\begin{aligned}
\iint_S \nabla \times \mathbf{F} \cdot d\mathbf{S} &= \int_0^{2\pi}\int_0^2 \left(\frac{r^2}{\sqrt{4-r^2}}(5\cos\theta + 2\sin\theta) + r\right) dr\,d\theta \\
&= \int_0^{2\pi} (5\cos\theta + 2\sin\theta)\,d\theta \int_0^2 \frac{r^2}{\sqrt{4-r^2}}\,dr + \int_0^{2\pi}\int_0^2 r\,dr\,d\theta \\
&= (5\sin\theta - 2\cos\theta)\Big|_0^{2\pi} \int_0^2 \frac{r^2}{\sqrt{4-r^2}}\,dr + 2\pi\int_0^2 r\,dr \\
&= 0\int_0^2 \frac{r^2}{\sqrt{4-r^2}}\,dr + 2\pi\left(\frac{r^2}{2}\right)\Big|_0^2 = 4\pi.
\end{aligned}$$

Next, calculate $\oint_{\partial S} \mathbf{F} \cdot d\mathbf{s}$. The appropriate orientation for ∂S is clockwise when viewed from the positive $\mathbf{k}$ direction. A parameterization is

$$\mathbf{x}(t) = (2\cos t,\ -2\sin t,\ 0),$$

where t goes from 0 to 2π, and the tangent vector is

$$\mathbf{x}'(t) = (-2\sin t,\ -2\cos t,\ 0).$$

The line integral comes out to

$$\begin{aligned}
\oint_{\partial S} \mathbf{F} \cdot d\mathbf{s} &= \int_{\mathbf{x}} (2y - z)\, dx + (x + y^2 - z)\, dy + (4y - 3x)\, dz \\
&= \int_0^{2\pi} 2(-2\sin t)(-2\sin t)\, dt + (2\cos t + 4\sin^2 t)(-2\cos t)\, dt \\
&= \int_0^{2\pi} \left(8\sin^2 t - 4\cos^2 t - 8\sin^2 t \cos t\right) dt \\
&= \int_0^{2\pi} (4(1 - \cos 2t) - 2(1 + \cos 2t))\, dt - 8\int_0^{2\pi} \sin^2 t \cos t\, dt \\
&= \int_0^{2\pi} (2 - 6\cos 2t)\, dt - 8\int_0^{2\pi} \sin^2 t \cos t\, dt \\
&= (2t - 3\sin 2t)\Big|_0^{2\pi} - \tfrac{8}{3}\sin^3 t\Big|_0^{2\pi} = 4\pi.
\end{aligned}$$

The content of Stokes' theorem is that these two integrals are equal.

10. As suggested, consider a vector field $\mathbf{F}(x, y, z) = M(x, y)\mathbf{i} + N(x, y)\mathbf{j}$, where M and N are scalar functions not depending on z. Suppose that D is a closed, bounded region in the xy-plane with boundary C and orient C so that D is on the left as as one traverses C. This is the correct orientation to apply Stokes' theorem if we choose the upward pointing normal for D. By Stokes' theorem,

$$\iint_D \nabla \times \mathbf{F} \cdot d\mathbf{S} = \oint_C \mathbf{F} \cdot d\mathbf{s} = \oint_C M\, dx + N\, dy. \tag{1}$$

Calculate the left hand side.

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M & N & 0 \end{vmatrix} = \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\mathbf{k}$$

(Remember that M and N do not depend on z, so $\frac{\partial N}{\partial z}$ and $\frac{\partial M}{\partial z}$ are 0.) Using x and y as parameters for D, the tangent vectors are $\mathbf{i}$ and $\mathbf{j}$, respectively, so the normal vector is $\mathbf{i} \times \mathbf{j} = \mathbf{k}$.

$$\begin{aligned}
\iint_D \nabla \times \mathbf{F} \cdot d\mathbf{S} &= \iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\mathbf{k} \cdot \mathbf{k}\, dx\, dy \\
&= \iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\, dy \qquad (2)
\end{aligned}$$

Combining equations (1) and (2) we get Green's theorem:

$$\oint_C M\, dx + N\, dy = \iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\, dy.$$

16. Let D be the region in the plane $2x - 3y + 5z = 17$ enclosed by the curve C. Since the plane is a level set of the function $f(x, y, z) = 2x - 3y + 5z$, we can get a normal vector by taking the gradient:

$$\mathbf{N} = \nabla f = 2\mathbf{i} - 3\mathbf{j} + 5\mathbf{k}.$$

Notice that $\mathbf{N}$ does not depend on x, y, or z. We want to compute $\oint_C \mathbf{F} \cdot d\mathbf{s}$, where $\mathbf{F}$ is the vector field

$$(3\cos x + z)\,\mathbf{i} + (5x - e^y)\,\mathbf{j} - 3y\,\mathbf{k}.$$

Using Stokes' theorem we get

$$\oint_C \mathbf{F} \cdot d\mathbf{s} = \iint_D \nabla \times \mathbf{F} \cdot d\mathbf{S}.$$

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3\cos x + z & 5x - e^y & -3y \end{vmatrix} = -3\mathbf{i} + \mathbf{j} + 5\mathbf{k}$$

The unit normal vector to D is $\pm\frac{\mathbf{N}}{\|\mathbf{N}\|}$; the sign is determined by the orientation of C. Putting it all together,

$$\begin{aligned} \oint_C \mathbf{F} \cdot d\mathbf{s} &= \iint_D \nabla \times \mathbf{F} \cdot d\mathbf{S} \\ &= \pm\frac{1}{\|\mathbf{N}\|} \iint_D (-3\mathbf{i} + \mathbf{j} + 5\mathbf{k}) \cdot (2\mathbf{i} - 3\mathbf{j} + 5\mathbf{k})\, dS \\ &= \pm\frac{1}{\|\mathbf{N}\|} \iint_D 16\, dS = \pm\frac{16}{\|\mathbf{N}\|}\,(\text{area of } D)\,. \end{aligned}$$

DELA 2.1

T/F

4. F

10. F

Problems

6. A is a 3×3 matrix. The entries given are

$$A = \begin{bmatrix} & -1 & 2 \\ & & 3 \\ & & \end{bmatrix}.$$

Using $a_{ji} = -a_{ij}$ we can get

$$A = \begin{bmatrix} & -1 & 2 \\ 1 & & 3 \\ -2 & -3 & \end{bmatrix}.$$

The entries on the main diagonal have $i = j$. For these elements, $a_{ii} = -a_{ii}$, which means that $a_{ii} = 0$. Thus, the whole matrix is

$$A = \begin{bmatrix} 0 & -1 & 2 \\ 1 & 0 & 3 \\ -2 & -3 & 0 \end{bmatrix}.$$

14. We assemble B by writing $\mathbf{b}_1$, $\mathbf{b}_2$, $\mathbf{b}_3$, and $\mathbf{b}_4$ in its columns:

$$B = \begin{bmatrix} 2 & 5 & 0 & 1 \\ -1 & 7 & 0 & 2 \\ 4 & -6 & 0 & 3 \end{bmatrix}.$$

The row vectors of B we get by reading horizontally:

$$[2\ \ 5\ \ 0\ \ 1], \qquad [-1\ \ 7\ \ 0\ \ 2], \qquad \text{and} \quad [4\ \ -6\ \ 0\ \ 3].$$

20. In order to be lower triangular, our matrix A needs zeros above the main diagonal.

$$A = \begin{bmatrix} & 0 & 0 \\ & & 0 \\ & & \end{bmatrix}$$

To make A skew-symmetric, the entries below the main diagonal should be the negatives of those above:

$$A = \begin{bmatrix} & 0 & 0 \\ 0 & & 0 \\ 0 & 0 & \end{bmatrix}$$

Finally, a skew-symmetric matrix needs zeros on the main diagonal, since $a_{ii} = -a_{ii}$.

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

So in fact this matrix of zeros is the *only* 3×3 lower triangular skew-symmetric matrix.

22. A good example of a function that hits the same value twice but not three times is a quadratic function. If $f(t) = t(t-1)$ then $f(0) = f(1) \neq f(2)$, since $f(0)$ and $f(1)$ are 0 and quadratic functions have no more than 2 roots. Any constant multiple of f will have this same property. So for A we could pick

$$A = \begin{bmatrix} t(t-1) & 2t(t-1) & 3t(t-1) \\ 4t(t-1) & 5t(t-1) & 6t(t-1) \\ 7t(t-1) & 8t(t-1) & 9t(t-1) \end{bmatrix}.$$

DELA 2.2

T/F

4. F

Problems

2. If $2A + B - 3C + 2D = A + 4C$, then we can rearrange to get

$$D = \tfrac{1}{2}(-A - B + 7C).$$

Now, substitute the given matrices for A, B, and C and simplify:

$$\begin{aligned}
D &= \tfrac{1}{2}(-A - B + 7C) \\
&= \frac{1}{2}\left(- \begin{bmatrix} 2 & -1 & 0 \\ 3 & 1 & 2 \\ -1 & 1 & 1 \end{bmatrix} - \begin{bmatrix} 1 & -1 & 2 \\ 3 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix} + 7 \begin{bmatrix} -1 & -1 & 1 \\ 1 & 2 & 3 \\ -1 & 1 & 0 \end{bmatrix} \right) \\
&= \frac{1}{2}\left(\begin{bmatrix} -2 & 1 & 0 \\ -3 & -1 & -2 \\ 1 & -1 & -1 \end{bmatrix} + \begin{bmatrix} -1 & 1 & -2 \\ -3 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix} + \begin{bmatrix} -7 & -7 & 7 \\ 7 & 14 & 21 \\ -7 & 7 & 0 \end{bmatrix} \right) \\
&= \frac{1}{2} \begin{bmatrix} -10 & -5 & 5 \\ 1 & 13 & 18 \\ -5 & 5 & -1 \end{bmatrix} = \begin{bmatrix} -5 & -\frac{5}{2} & \frac{5}{2} \\ \frac{1}{2} & \frac{13}{2} & 9 \\ -\frac{5}{2} & \frac{5}{2} & -\frac{1}{2} \end{bmatrix}
\end{aligned}$$

10. The vector $A\mathbf{c}$ is a linear combination of the column vectors of A. The coefficients of this linear combination are the entries of $\mathbf{c}$. The column vectors of A are

$$\begin{bmatrix} 3 \\ 2 \\ 7 \end{bmatrix}, \qquad \begin{bmatrix} -1 \\ 1 \\ -6 \end{bmatrix}, \qquad \text{and} \qquad \begin{bmatrix} 4 \\ 5 \\ 3 \end{bmatrix}.$$

Now we can compute

$$\begin{aligned}
A\mathbf{c} &= 2 \begin{bmatrix} 3 \\ 2 \\ 7 \end{bmatrix} + 3 \begin{bmatrix} -1 \\ 1 \\ -6 \end{bmatrix} - 4 \begin{bmatrix} 4 \\ 5 \\ 3 \end{bmatrix} \\
&= \begin{bmatrix} 6 \\ 4 \\ 14 \end{bmatrix} + \begin{bmatrix} -3 \\ 3 \\ -18 \end{bmatrix} + \begin{bmatrix} -16 \\ -20 \\ -12 \end{bmatrix} \\
&= \begin{bmatrix} -13 \\ -13 \\ -16 \end{bmatrix}.
\end{aligned}$$

24. If A and C are $m \times n$ matrices, we aim to prove that

$$\left(A^T\right)^T = A \tag{1}$$

and

$$(A + C)^T = A^T + C^T. \tag{2}$$

Let a_{ij} be the entries of the matrix A, with $1 \leq i \leq m$ and $1 \leq j \leq n$, and let c_{ij} be the entries of C.

To prove (1), let $B = A^T$. Since A is $m \times n$, it follows that B is $n \times m$. If b_{ij} are the entries of B, with $1 \le i \le n$ and $1 \le j \le m$, then $b_{ij} = a_{ji}$. Now, let $D = B^T$, so that $D = \left(A^T\right)^T$. The matrix B is $n \times m$, so D will be $m \times n$. The entries of D are $d_{ij} = b_{ji}$, with $1 \le i \le m$ and $1 \le j \le n$. The dimensions of D and A are the same—both are $m \times n$ matrices. Furthermore, $d_{ij} = b_{ji} = a_{ij}$, so we can conclude that $D = A$, that is, we have proved (1).

Let's move on to (2). If $E = A + C$ and e_{ij} are its entries then $e_{ij} = a_{ij} + c_{ij}$ for $1 \le i \le m$ and $1 \le j \le n$. Its transpose is the $n \times m$ matrix with entries e_{ji}. On the other hand, the entries of A^T and C^T are a_{ji} and c_{ji}, respectively, so the entries of $F = A^T + C^T$ are $f_{ij} = a_{ji} + c_{ji}$. Since A and C are $m \times n$ matrices, A^T, C^T and F are $n \times m$. For a start the dimensions of E^T agree with those of F. As for the entries, $f_{ij} = a_{ji} + c_{ji} = e_{ji}$, so we conclude that $F = E^T$, which proves (2).

36. (a) In order for AA^T to be symmetric we need $\left(AA^T\right)^T = AA^T$. Well, part (3) of Theorem 2.2.21 tells us that

$$\left(AA^T\right)^T = \left(A^T\right)^T A^T$$

and then we can use part (1) to get

$$\left(A^T\right)^T A^T = AA^T.$$

(b) To show $(ABC)^T = C^T B^T C^T$, use part (3) of Theorem 2.2.21 twice:

$$(ABC)^T = C^T (AB)^T = C^T B^T A^T.$$

38. To differentiate a matrix function, take the derivative of each entry:

$$\begin{aligned} \frac{d}{dt} t &= 1, \\ \frac{d}{dt} \sin t &= \cos t, \\ \frac{d}{dt} \cos t &= -\sin t, \text{ and} \\ \frac{d}{dt} 4t &= 4. \end{aligned}$$

Thus,

$$\frac{dA}{dt} = \begin{bmatrix} 1 & \cos t \\ -\sin t & 4 \end{bmatrix}.$$

42. We integrate each entry of the matrix function:

$$\int_0^{\frac{\pi}{2}} \cos t \, dt = \sin t \Big|_0^{\frac{\pi}{2}} = 1 \quad \text{and}$$

$$\int_0^{\frac{\pi}{2}} \sin t \, dt = -\cos t \Big|_0^{\frac{\pi}{2}} = 1,$$

so the integral of A is

$$\int_0^{\frac{\pi}{2}} A(t) \, dt = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$