5.6 Euler Paths and Cycles

One of the oldest and most beautiful questions in graph theory originates from a simple challenge that can be played by children. The town of Königsberg (now

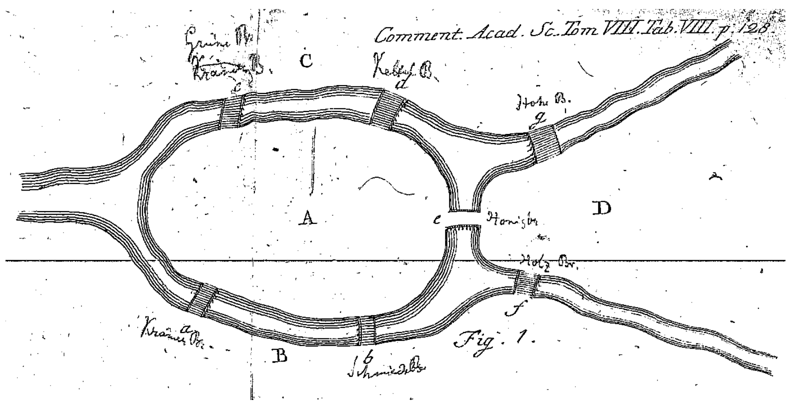

Figure 33: An illustration from Euler's 1741 paper on the subject.

Kaliningrad, Russia) is situated near the Pregel River. Residents wondered whether they could they begin a walk in one part of the city and cross each bridge exactly once. Many tried and many failed to find such a path, though understanding why such a path cannot exist eluded them.

Graph theory, which studies points and connections between them, is the perfect setting in which to study this question. Land masses can be represented as vertices of a graph, and bridges can be represented as edges between them. Generalizing the question of the Königsberg residents, we might ask whether for a given graph we can "travel" along each of its edges exactly once. Euler's work elegantly explained why in some graphs such trips are possible and why in some they are not.

Definition 23. *A* **path** *in a graph is a sequence of adjacent edges, such that consecutive edges meet at shared vertices. A path that begins and ends on the same vertex is called a* **cycle**.

Note that every cycle is also a path, but that most paths are not cycles. Figure 34 illustrates K_5, the complete graph on 5 vertices, with four different paths highlighted; Figure 35 also illustrates K_5, though now all highlighted paths are also cycles.

In some graphs, it is possible to construct a path or cycle that includes every edges in the graph. This special kind of path or cycle motivate the following definition:

Definition 24. *An* **Euler path** *in a graph G is a path that includes every edge in G; an* **Euler cycle** *is a cycle that includes every edge.*

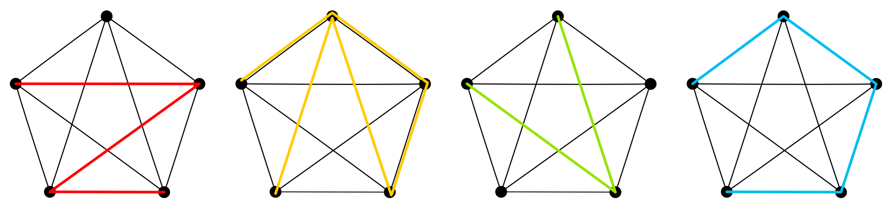

Figure 34: K_5 with paths of different lengths.

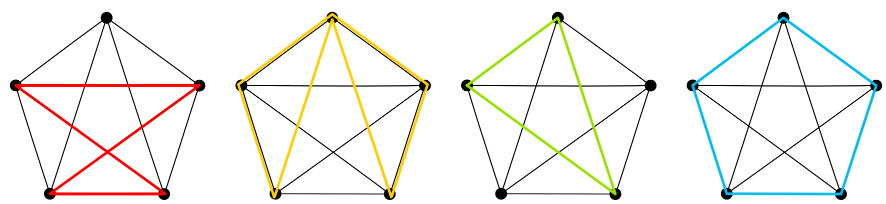

Figure 35: K_5 with cycles of different lengths.

Spend a moment to consider whether the graph K_5 contains an Euler path or cycle. That is, is it possible to travel along the edges and trace each edge exactly one time. It turns out that it is possible. One way to do this is to trace the (five) edges along the boundary, and then trace the star on the inside. In such a manner one travels along each of the ten edges exactly one time. One also ends at the same point at which one began, and so this Euler path is also an Euler cycle.

This example might lead the reader to mistakenly believe that every graph in fact has an Euler path or Euler cycle. It turns out, however, that this is far from true. In particular, Euler, the great 18th century Swiss mathematician and scientist, proved the following theorem.

Theorem 13. *A connected graph has an Euler cycle if and only if all vertices have even degree.*

This theorem, with its "if and only if" clause, makes two statements. One statement is that if every vertex of a connected graph has an even degree then it contains an Euler cycle. It also makes the statement that only such graphs can have an Euler cycle. In other words, if some vertices have odd degree, the the graph cannot have an Euler cycle. Notice that this statement is about Euler cycles and not Euler paths; we will later explain when a graph can have an Euler path that is not an Euler cycle.

Proof. How can show that every graph with an Euler cycle has no vertices with odd degree? One way to do this is to imagine starting from a graph with no edges, and "traveling" along the Euler cycle, laying down edges one at a time, until we have constructed our original graph. We consider what happens to the degree of each vertex as we travel around the graph adding edges. Notice that before doing any traveling, and so before we draw in any of the edges, the degree of each vertex is 0. Let us now consider the vertex from which we start and call it v_0. After leaving v_0 and laying down the first edge, we have increased the degree of v_0 by 1, i.e., $\deg(v_0) = 1$. So long as we don't return to v_0, its degree will stay 1. Now, notice what happens as we travel along our graph adding edges. Every time we pass through a vertex, we increase its degree by 2. The reason for this is that every time we pass through a vertex, we add one degree for the edge "entering" it and one degree for the edge "exiting" it. The

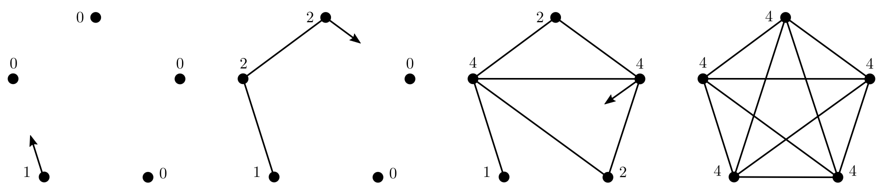

Figure 36: "Traveling" along an Euler cycle in K_5; numbers indicate vertex degrees at each point in "time".

definition of an Euler cycle requires that we end where we began, and so the final edge takes us to v_0, finally increasing its degree by exactly 1 and making it even again. At this point, the degrees of all vertices, including the single vertex from which we began and at which we ended, are all even. Figure 36 illustrates traveling along the edges of K_5 to construct an Euler cycle. $\square$

The above proof only shows that if a graph has an Euler cycle, then all of its vertices must have even degree. It does not, however, show that if all vertices of a (connected) graph have even degrees then it must have an Euler cycle. The proof for this second part of Euler's theorem is more complicated, and can be found in most introductory textbooks on graph theory.

Our proof above might motivate us to think about what happens if the Euler path we are considering is not a cycle. In other words, what happens if we travel along every edge in a graph but do not return to our starting point? Notice that the degree of the starting point v_0 will then remain odd, as will the last vertex which we visited, since we "entered" it but never "exited" it. Can any of the other vertices in the graph have odd degree? No, because all degrees began at 0, and only increased by 2 when they were visited, with the exception of the vertex from which we began the vertex on which we stopped. Therefore, all vertices have even degree with the exception of two, on which an Euler path begins and ends. This proves a second theorem, one about Euler paths:

Theorem 14. *A graph with more than two odd-degree vertices has no Euler path.*

Hamiltonian Paths and Cycles

Until now we have considered paths and cycles that can visit vertices multiple times. What happens if we require that a path visit every vertex exactly one time? Such paths and cycles are called Hamiltonian paths and Hamiltonian cycles, are the subject of much research. Suppose you would like to fly to every major airport in the continental US without visiting any of them more than once. Is this possible? Despite the similarity of this question to questions we considered in discussing Euler paths and cycles, considerably less is known about them. In particular, there seems to be no good way to look at a particular graph and know whether a Hamiltonian path or cycle exists, without trail and error.

Although less is known about Hamiltonian paths than about Euler paths, many things are known. In particular, it is known that all five of the Platonic solids have Hamiltonian cycles. Furthermore, every complete graph K_n also has a Hamiltonian cycle (can you see why?). In 1952, Gabriel Dirac proved that every (simple) graph on n vertices has a Hamiltonian cycle if the degree of every vertex is $n/2$ or greater. Although this theorem guarantees a Hamiltonian cycle under certain conditions, this does not mean that if a graph has a Hamiltonian cycle, then it must satisfy this condition.