5.4 Polyhedral Graphs and the Platonic Solids

Regular Polygons

In this section we will see how Euler's formula – unquestionably the most important theorem about planar graphs – can help us understand polyhedra and a special family of polyhedra called the Platonic solids. You might recall that *polygons* are two dimensional shapes such as triangles, rectangles, pentagons, and hexagons. Below are illustrated polygons with 3, 4, 5, and 6 edges. Each

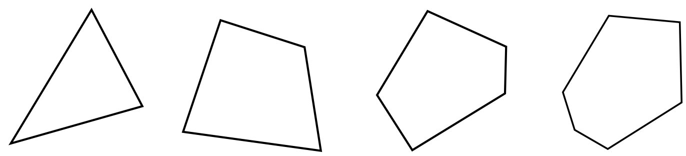


Figure 22: Polygons with 3, 4, 5, and 6 edges.

of these shapes is constructed using three or more straight line segments connected together at their endpoints. The lengths of these line segments are not necessarily the same for each side of the polygon, nor are the internal angles at which pairs of edges meet. Such shapes are called *irregular polygons*, or just *polygons*.

If all edges have the same length, and all pairs of edges meet at identical angles, then such shapes are called *regular polygons*. Regular polygons with 3, 4, 5, and 6 edges are illustrated below. Although most regular polygons

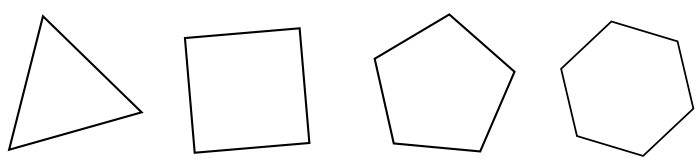


Figure 23: Regular polygons with 3, 4, 5, and 6 edges.

we encounter do not have many sides, we can construct a regular polygon for any number of sides greater than 2. The figure below illustrates several regular polygons with large numbers of edges. Although it might be difficult to construct

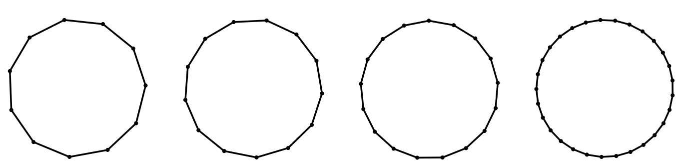


Figure 24: Regular polygons with 11, 13, 17, and 29 edges; small circles placed at corners to make edges more visible.

or count its number of edges, we can construct regular polygons with a hundred, a thousand, or even a million edges.

Polyhedra

Polyhedra are three-dimensional analogues of polygons. Instead of being constructed of line segments, polyhedra are constructed from polygons connected together along edges. The polygons do not necessarily need to be regular. Below are illustrated several polyhedra composed of different numbers of polygonal faces. The reader might notice that the first and third example stand out as

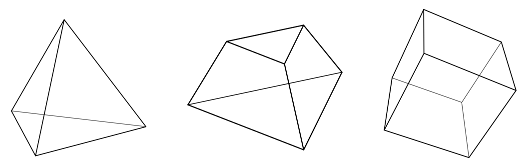


Figure 25: Several familiar polyhedra with 4, 5, and 6 faces.

being particularly symmetric. Indeed, in these two cases, each face of the polyhedron is an identical regular polygon. Moreover, the same number of faces meet at every corner. Such polyhedra are called *regular*, and can be considered three-dimensional analogues of two-dimensional regular polygons.

Despite some similarities between regular polygons and regular polyhedra, there turns out to be a fundamental difference in how many of them we can find. In particular, although for every $n \geq 3$ there exists a regular polygon with n sides, there are only five values of n for which there exists a regular polyhedron with n faces. These are known as the Platonic solids, and Euler's theorem will help us enumerate their possibilities.

Polyhedral Graphs

In order to make Euler's theorem useful in studying polyhedra, we need to understand the relationship between polyhedra and planar graphs. We begin by noting that every polyhedron uniquely determines a graph up to isomorphism. To see this, we place a vertex at every corner of a polyhedron. If we consider the cube, for example, we can construct a graph that has 8 vertices, one corresponding to each corner. Next, we connect pairs of vertices if both lie along the same edge in the polyhedron. Graphs constructed in this manner are called **polyhedral graphs**. If we wrote out this graph in terms of its vertex set V and edge set E, we would have:

$$\begin{aligned} V &= \{a, b, c, d, e, f, g, h\} \\ E &= \{\{a, b\}, \{b, c\}, \{c, d\}, \{d, a\}, \\ &\quad \{e, f\}, \{f, g\}, \{g, h\}, \{h, e\}, \\ &\quad \{a, e\}, \{b, f\}, \{c, g\}, \{d, h\}\}. \end{aligned}$$

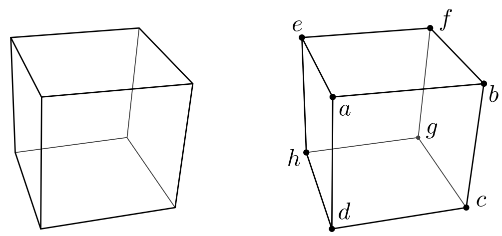


Figure 26: A cube and its associated graph.

Not only can we construct a graph using the corners and edges of a polyhedron, but all such graphs turn out to be planar graphs – this is a subtle point worth considering before continuing. One way to think about why this is true is by considering what happens when we choose one face of the polyhedron and "stretch it out"; we call this face the "outside" face for reasons that will become clear in a moment. For example, in the example above, consider what happens when we take the face *abcd* and stretch it out. In this example, we can move

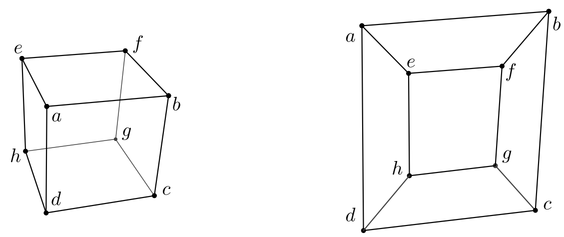


Figure 27: Moving vertices of the polyhedral graph to illustrate that it is planar; we "stretched out" the vertices a, b, c, and d, and "moved" all remaining vertices inside it.

out the vertices a, b, c, and d, and move in the remaining vertices. Since this graph is now drawn without any edges crossing one another, it is clear that the graph associated with the cube is indeed planar. The face *abcd* is now drawn on the outside of the graph, thus justifying its name. We could have chosen any one of the six faces to be the outside face, though in each case the planar graph we drew would have looked the same.

While in the case above, which face we choose to be the "outside" face makes no difference to the picture of the planar graph (aside from vertex labelings), this is not generally the case. Consider for example the triangular prism, illustrated below with an associate polyhedral graph. As a polyhedral graph, we can draw this is in several ways. If we choose one of the triangular faces, such as *abc* as the outside face, we will obtain a drawing such as the one on the left of Figure 29; if we choose a quadrilateral face, such as *abed*, then we will obtain a drawing such as that on the right. These graphs are identical as graphs.

Although the graph associated with every polyhedron is a planar graph,

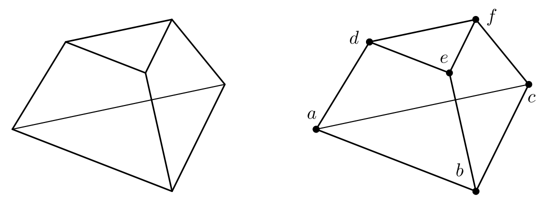


Figure 28: Triangular prism and an associated polyhedral graph.

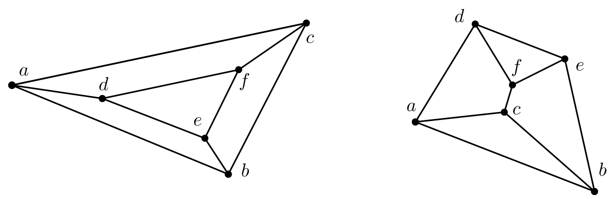


Figure 29: Two planar drawings of the polyhedral graph associated with the triangular prism.

there exist planar graphs that are not graphs of polyhedra. Although we will not consider examples of such here, it is a point worth thinking about. Can you think of a planar graph that is not the graph of any polyhedron?

The Platonic Solids

Euler's formula allows us to use what we know about planar graphs to prove that there exist only five regular polyhedra. For our purposes, we consider the following definition:

Definition 22. *A* **regular polyhedron** *is one in which all faces are identical regular polygons, and such that the same number of faces meet at every corner.*

In terms of planar graphs, this means that every face in the planar graph (including the outside one) has the same degree (number of edges on its boundary), and every vertex has the same degree. We are now able to prove the following theorem.

Theorem 10. *There are no more than 5 regular polyhedra.*

Proof. In proving this theorem we will use n to refer to the number of edges of each face of a particular regular polyhedron, and d to refer to the degree of each vertex. We will show that there are only five different ways to assign values to n and d that satisfy Euler's formula for planar graphs.

Let us begin by restating Euler's formula for planar graphs. In particular:

$$v - e + f = 2. \tag{48}$$

In this equation, v, e, and f indicate the number of vertices, edges, and faces of the graph. Previously we saw that if we add up the degrees of all vertices in a graph (not necessarily planar) we obtain a number that is twice the number of edges. In equation form, we have

$$\sum_{i}^{v} \deg(v_i) = 2e,$$

where v_i is the ith vertex. In our case, since we are only considering graphs in which each vertex has the same degree d, we can rewrite this as $vd = 2e$, or

$$v = 2e/d. \tag{49}$$

We have also seen that if we add up the degrees of all faces in a planar graph we obtain a number that is twice the number of edges. In equation form, we have

$$\sum_{i}^{f} \deg(f_i) = 2e,$$

where f_i is the ith face. In our case, since we are only considering graphs in which each face has the same degree n, we can rewrite this as $fn = 2e$, or

$$f = 2e/n. \tag{50}$$

Equations 48, 49, and 50 provide us with sufficient information to prove that there are at most five regular polyhedra. First, we can substitute Equations 49 and 50 into Equation 48 to obtain:

$$\frac{2e}{d} - e + \frac{2e}{n} = 2. \tag{51}$$

We can rearrange the terms and divide all of them by $2e$ to obtain

$$\frac{1}{n} + \frac{1}{d} = \frac{1}{2} + \frac{1}{e}. \tag{52}$$

Since the number of edges in a polyhedral graph is always positive, the term $\frac{1}{e}$ must be positive. Therefore, if we remove the term $\frac{1}{e}$ from the right-hand side, we make it smaller than the left-hand side.

$$\frac{1}{n} + \frac{1}{d} > \frac{1}{2}. \tag{53}$$

This inequality, which must be true for every regular polyhedral graph, tells us about the possible values of n and d. First, notice that if n and d are both very large, then the left-hand side will be very small. For example, notice that if $n = 4$ and $d = 4$, then we obtain the false inequality:

$$\frac{1}{4} + \frac{1}{4} > \frac{1}{2}. \tag{54}$$

Since this is not true, at least one of n and d must be smaller than 4. However, notice also that neither of n or d can be smaller than 3, since faces cannot have fewer than 3 edges, and vertices, in polyhedral graphs, cannot have degree smaller than 3 (think about this). Therefore, for any regular polyhedron, at least one of n or d must be exactly 3.

Let us consider each of the two cases individually. We begin with $n = 3$, or polyhedral graphs in which all faces have three edges, i.e., all faces are triangles. Substituting $n = 3$ into Equation 53, we find out that $\frac{1}{d} > \frac{1}{6}$, or that $d < 6$. This leaves us with three options, either $d = 3$, 4, or 5. This gives us three possible regular polyhedra entirely of triangles.

Alternatively, we consider what happens when we require that $d = 3$. Similar calculations show that this forces n to be either 3, 4, or 5. This too gives us three possible regular polyhedra, all of whose vertices have degree 3. Notice that there is one overlap between the two sets of polyhedra – the one all of whose faces are triangles and all of whose vertices have degree 3. Therefore, there are only five unique pairs of n and d that can describe regular polyhedra.

Each of these five choices of n and d results in a different regular polyhedron, illustrated below. $\square$

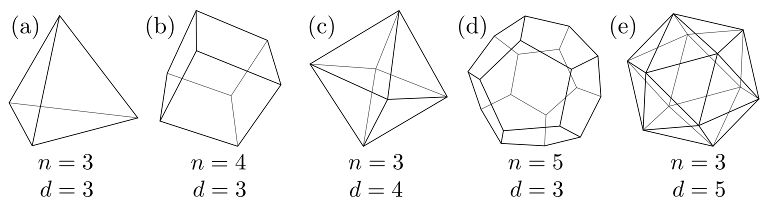


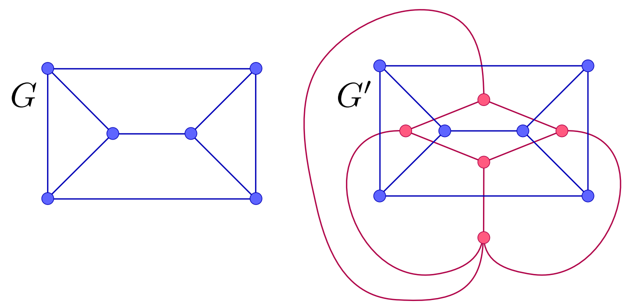
Figure 30: The five regular polyhedra, also known as the Platonic solids. Below are listed the numbers of vertices v, edges e, and faces f of each regular polyhedron, as well as the number of edges per face n and degree d of each vertex.

Name of Polyhedron	v	e	f	d	n
(a) tetrahedron	4	6	4	3	3
(b) cube	8	12	6	3	4
(c) octahedron	6	12	8	4	3
(d) dodecahedron	20	30	12	3	5
(e) icosahedron	12	30	20	5	3

Dual Graphs and Dual Polyhedra

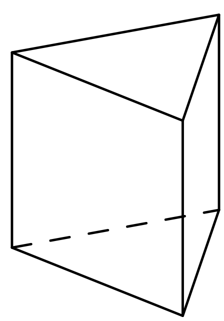
A beautiful topic that arises in many areas of pure and applied mathematics is that of *dual* objects or problems. That is, we might be given a particular problem or object, and to study it, it is best to study another problem that is intimately related to the first problem. Here we briefly consider dual graphs and dual polyhedra.

Think for a moment about how we might begin with a planar graph and use it to generate a new one. In particular, think about what happens when we take a planar graph G and use it to make a new graph G' (pronounced G prime) in the following manner. For every face in G we make a vertex in the new graph G'. We then connect pairs of vertices in G' if corresponding faces in G share an edge.



Above to the left is illustrated a graph G with five faces, including the outside face; this graph is colored blue. We make a new graph that we call G' based on G, as described above. Notice that for every face with degree 3 in G there is a vertex with degree 3 in G', and for every face with degree 4 in G there is a vertex with degree 4 in G'. Notice also that we create one edge in G' for every edge in G.

We could have also performed this construction on polyhedra instead of on planar representations of them. For example, consider the triangular prism



illustrated above. Indeed, this graph is isomorphic to G above. We can construct a new polyhedra by placing one vertex at the center of each face, and then connected vertices whose corresponding faces share an edge. This is called the dual polyhedra. What would it look like?

[Notes here are incomplete; additional information on this subject can be found on https://en.wikipedia.org/wiki/Dual_graph]