Map Colorings

Last time we considered an application of graph theory for studying polyhedra. In particular, we used Euler's formula to prove that there can be no more than five regular polyhedra, which are known as the Platonic Solids. Many classical philosophers believed in a mystical correspondence between these polyhedra and air, earth, fire, and water – which they understood to be the four basic elements of the world; the fifth polyhedron corresponds to the universe itself, or to the 'aether'. This belief is no longer widespread, but it might be of some historical or cultural interest to some readers.

Today we consider an application of graph theory, and of Euler's formula, in studying the problem of how maps can be colored. Map-makers often color adjacent geo-political regions differently, so that map-readers can easily distinguish distinct regions. In the illustration below on the left, we color Pennsylvania orange, West Virginia yellow, New York purple, and so forth. If we had a box of 64 Crayola crayons, of course we would have enough colors so that every state could have a distinct color. But if we only have five or six colors, we certainly can't color every state a different color. But maybe we can still color every *adjacent* state a different color. Is that possible? Or, to make this question a bit more precise – how many colors would we need to make sure that adjacent states never share a color? This is a classical problem in graph theory, and in this section we'll use graph theory, and in particular planar graphs and Euler's formula, to study it.

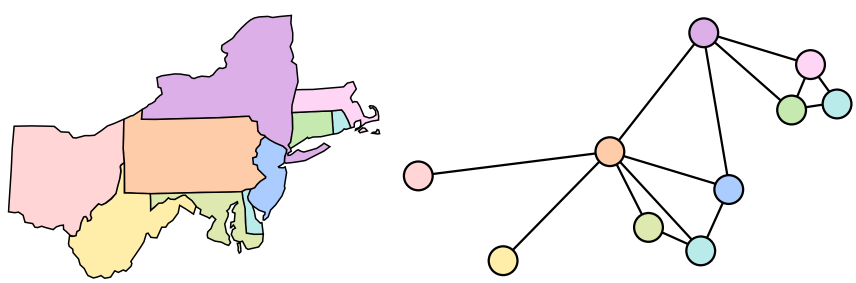

Figure 2: Map of several northeastern states, and a representation of this map as a planar graph.

To see how graphs can be relevant to studying maps, we construct a new graph such that each state is represented by a vertex, and such that two vertices are connected by an edge if and only if the two states share a boundary. The illustration above on the right shows such a graph. Although it is not entirely obvious, such a graph is always planar, as crossing edges would indicate crossing borders between adjacent states, which cannot occur.

Since maps can be represented as planar graphs, if we can prove that some number of colors is always sufficient to color the vertices of a planar graph, then we can also know that that number of colors is sufficient to color a map. In what follows, we will prove that 6 colors is always sufficient to color a planar graph. In fact, 4 colors is also sufficient to color any planar graph, but the proof for that statement is substantially more involved, and would take the remainder of the semester to investigate. The two papers that proved this theorem in 1976 required well over a hundred pages to prove this, and are well beyond the treatment here. However, we can still prove that 6 colors are sufficient. Proving that 5 colors is also sufficient is also possible, and might be covered later in this section.

Vertices with Small Degree

In order to prove that every planar graph can be colored with 6 colors, we first need to prove the following theorem:

Theorem 1. *Every planar graph contains at least one vertex with degree at most 5.*

Proof. We have previously seen that for an arbitrary graph with v vertices and e edges, we can calculate e by considering the degrees of all vertices v_i. In particular,

$$\sum_{i=1}^{v} \deg(v_i) = 2e. \tag{1}$$

We can use this relationship to write the average degree of a vertex in terms of e and v. If we take the sum of the degrees and divide that sum by the total number of vertices, then we can write this average, which we call $\overline{\deg}$, as:

$$\overline{\deg} = 2e/v. \tag{2}$$

Now we consider another similar relationship, but instead of considering the degrees of vertices, we consider the number of edges of the faces. If we take into account the "outside" face of a planar graph, then every edge in a planar graph appears in exactly two faces. If a graph has f faces, and if we use f_i to refer to the number of edges around face i, then we can write:

$$\sum_{i=1}^{f} f_i = 2e. \tag{3}$$

Since every face must have at least 3 faces (i.e., $f_i \geq 3$ for all i), then we can rewrite Equation 3 as an inequality

$$3f \leq 2e, \tag{4}$$

or equivalently as

$$f \leq \frac{2}{3}e. \tag{5}$$

We should recall here Euler's formula for planar graphs which can be written as:

$$f = 2 + e - v. \tag{6}$$

Combining the previous two equations, we have:

$$2 + e - v \leq \frac{2}{3}e. \tag{7}$$

Basic high-school algebra allows us to rearrange this and conclude that:

$$2e/v \leq 6 - \frac{12}{v}. \tag{8}$$

Of course, we have already seen above that this quantity $2e/v$ is merely the average degree of a vertex. In other words, we have:

$$\overline{\text{deg}} \leq 6 - \frac{12}{v}. \tag{9}$$

It is clear that the right-hand side of Equation 9 is a number smaller than 6. But if the degree of every vertex was 6 or greater, then this average would be 6 or larger. Therefore we can conclude that every planar graph must have at least one vertex with degree at most 5. □

Every Planar Graph is 6-colorable

Knowing that every planar graph has at least one vertex with degree at most 5 allows us to quickly prove that:

Theorem 2. *The vertices of every planar graph can be colored using 6 colors in such a way that no pair of vertices connected by an edge share the same color.*

Proof. We begin by noting that every graph on 6 or fewer vertices can, of course, be colored with 6 colors, since we can color every vertex using a different color. We can then consider $v = 6$ the *base case* for our induction proof.

Next we show that if the above is true for all planar graphs with $v = k$ vertices, then it must also be true for all planar graphs with $v = k + 1$ vertices. Let us consider a graph G on $v = k+1$ vertices; part of such a graph is illustrated in Figure 5. We want to show that G can be colored using at most 6 colors.

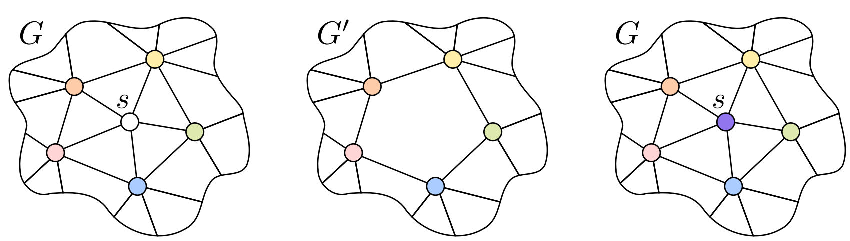

Figure 3: Parts of a graph G with $v = k + 1$ vertices, and of a modified version, which we call G', obtained by removing the vertex s. After putting back s, it is clear that we can color it using a color not used by any of its neighbors.

From Theorem 1 we know that G must have at least one vertex with degree at most 5. Let us find one of those vertices and call it s. For a moment, let's consider what happens when we remove vertex s. We construct a new graph which is identical to G except that s is now removed; we call this new graph G', to indicate that it's a modified version of G. Since G had $v = k + 1$ vertices, of course G' has $v = k$ vertices, since we removed one vertex. By the induction hypothesis, all graphs with $v = k$ vertices can be colored with 6 colors, so G', which has $v = k$ vertices, can be colored with only 6 colors.

Let's now look at what happens when we put vertex s back in to obtain our original graph G. In other words, we put s back into where it was originally. Since s had at most 5 neighbors, of course we can put it back in and color it a new color that is not one of those five colors. This shows that if all planar graphs with $v = k$ vertices can be colored using 6 colors, then so can all planar graphs with $v = k + 1$ vertices, completing our proof by induction. $\square$

It is also possible to prove, in a reasonable short amount of space, that every planar graph can be colored with only 5 colors. We will leave this proof for another time. As we have noted earlier, it is actually true that every map can be colored using only 4 colors, but the proof of that statement is very complicated and well beyond the tools we have developed so far.