Extra Credit Problem

In this problem we will derive Kepler's first law!
The first law states the following:

Let $\vec{r}(t)$ be a solution of the basic equation of planetary motion. This solution is either an ellipse which has one focal point at the Sun, a branch of a hyperbola which has one focal point at the Sun, a parabola whose focal point is the Sun, or an open ray emanating from the Sun.

Recall that in class we set up the polar coordinate system corresponding to a motion and we had

$$\vec{r}(t) = r\vec{u}_r$$

$$\vec{r}(t)' = r'\vec{u}_r + r\theta'\vec{u}_\theta$$

$$\vec{r}(t)'' = (r'' - r\theta'^2)\vec{u}_r + (2r'\theta' + r\theta'')\vec{u}_\theta$$

We also showed in class that $\vec{r}(t) \times \vec{r}(t)'$ is a constant vector. (The notation ' means that we differentiate with respect to t.)

- Show that this implies that $r^2\theta'$ is a constant, denote the constant by A.
- We also know that the gravitational force is a constant multiple of $-\frac{\vec{r}(t)}{r^3} = -\frac{\vec{u}_r}{r^2}$.
- Conclude that $r'' - r'\theta'^2$ is a constant multiple of $-\frac{1}{r^2}$, denote that constant by B, and moreover that $2r'\theta' + r\theta'' = 0$.
- Prove that the following quantity

  $$E = r'^2 + r^2\theta'^2 - 2\frac{B}{r}$$

  is constant.
- Use that $r^2\theta' = A$ to show that r' satisfies the following equation

  $$r'^2 = E + 2\frac{B}{r} - \frac{A^2}{r^2}.$$

- This is hard to integrate. Rather look at

  $$\frac{d\theta}{dr} = \frac{d\theta}{dt}\frac{dt}{dr} = \frac{\theta'}{r'} = \frac{A/r^2}{\sqrt{E + 2\frac{B}{r} - \frac{A^2}{r^2}}}$$

  (we can assume that $\frac{d\theta}{dr}$ is positive)

- Finally, integrate, and show that r satisfies the following equation

$$r = \frac{C}{1 + E\cos(\theta - \theta_0)}$$

  for some constants C, E and θ_0.

- Show that depending on the constants we get the conic sections stated in Kepler's First Law. (you can assume that $\theta_0 = 0$)