Practice midterm:

1) a) Let $x = my$, then:

$$\frac{xy}{x^2+y^2} = \frac{my^2}{m^2y^2+y^2} = \frac{m}{m^2+1}, \text{ so LDE.}$$

b) $|\sin y| \leq |y|$, therefore

$$\frac{x^2 \sin y}{x^2+y^2} \leq \frac{x^2|y|}{x^2+y^2} \leq |y|,$$

similarly,

$$\frac{x^2 \sin y}{x^2+y^2} \geq -|y|.$$

Since $y \to 0$, thus $\frac{x^2 \sin y}{x^2+y^2} \to 0$.

c) Same idea:

$$-|x| \leq \frac{x^5}{x^4+y^4} \leq |x|, \text{ thus } \frac{x^5}{x^4+y^4} \to 0 \text{ as } (x,y) \to (0,0).$$

d) $\frac{x^3+y^3}{x^3-y^3}$, let $x = my$, then $\frac{m^3y^3+y^3}{m^3y^3-y^3} = \frac{m^3+1}{m^3-1}$,

$$\frac{m^3y^3+y^3}{m^3y^3-y^3} = \frac{x^3+y^3}{x^3-y^3}$$

thus LDE.

e) $$\frac{x-y}{x^3-y^3} = \frac{x-y}{(x-y)(x^2+xy+y^2)} = \frac{1}{x^2+xy+y^2} = \frac{2}{(x+y)^2+x^2+y^2}$$

The limit is $+\infty$.

2, $\frac{\partial f}{\partial x} = 6xy - 6x$ $\qquad$ $\frac{\partial f}{\partial y} = 3x^2 + 3y^2 - 6y.$

if $\frac{\partial f}{\partial x} = 6xy - 6x = 0$, then $\underline{x = 0}$ or $6y = 6 \Rightarrow \underline{y = 1}$

if $x = 0$ and $\frac{\partial f}{\partial y} = 0$, then $3y^2 - 6y = 0$ so $y = 0$ or $2$.

if $y = 1$ and $\frac{\partial f}{\partial y} = 0$ then $3x^2 - 3 = 0$ so $x = 1$ or $x = -1$.

So 4 critical pts:

$(0,0)$; $(0,2)$; $(1,1)$; $(-1,1)$.

Hessian:

$$\begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix} = \begin{pmatrix} 6y-6 & 6x \\ 6x & 6y-6 \end{pmatrix}$$

At $(0,0)$: $\begin{pmatrix} -6 & 0 \\ 0 & -6 \end{pmatrix} \longrightarrow$ max

$(0,2)$: $\begin{pmatrix} 6 & 0 \\ 0 & 6 \end{pmatrix} \longrightarrow$ min

$(1,1)$: $\begin{pmatrix} 0 & 6 \\ 6 & 0 \end{pmatrix} \longrightarrow$ saddle

$(-1,1)$. $\begin{pmatrix} 0 & -6 \\ -6 & 0 \end{pmatrix} \longrightarrow$ saddle

$$\frac{\partial f}{\partial x} = y + 2x \qquad \frac{\partial f}{\partial y} = x - 2y$$

if $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0 \Rightarrow$ $y + 2x = 0 \rightarrow y = -2x$ $\quad y = 0.$

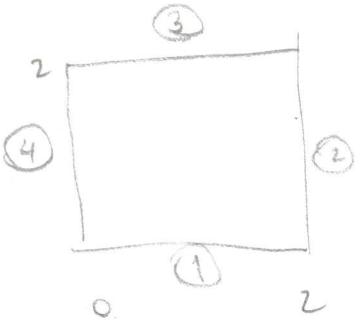
$x - 2y = 0$

$x - 2(-2x) = 0$

$5x = 0 \Rightarrow x = 0$

Only critical pt: $(0,0)$.

Boundary:




1: $y = 0 \quad 0 \le x \le 2$:

$f(x,0) = x^2$. Its minimum $x = 0$ ;

maximum $x = 2$.

2 $\quad x = 2 \quad 0 \le y \le 2$

$f(2,y) = 2y + 4 - y^2 = -(1-y)^2 + 5$

minimum: $y = 0$ or $2$

maximum: $y = 1$:

3 $\quad y = 2 \quad 0 \le x \le 2$

$f(x,2) = 2x + x^2 - 4 = (1+x)^2 - 5$

minimum: $x = 0$

maximum: $x = 2$

4 $\quad x = 0 \quad 0 \le y \le 2$

$f(0,y) = -y^2$

min : $y = 2$

max $\quad y = 0$

Possibilities:

| pt | $(0,0)$ | $(2,2)$ $(2,0)$ | $(2,1)$ | $(0,2)$ |
|---|---|---|---|---|
| Value | 0 | 4 | 5 | $-4$ |

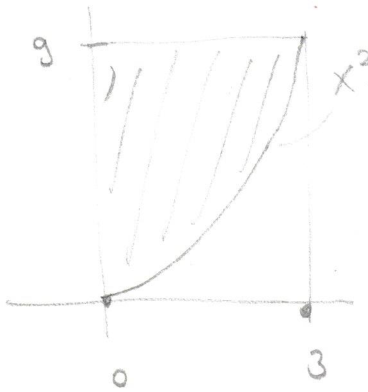
Thus max is 5

min is $-4$.

a) $$\int_0^1 \int_0^\pi x\cos(xy)\,dx\,dy = \int_0^\pi \int_0^1 x\cos(xy)\,dy\,dx = \int_0^\pi \sin(xy)\Big|_{y=0}^{1} dx$$

$$= \int_0^\pi (\sin x - \sin 0)\,dx = -\cos x\Big|_{x=0}^{x=\pi} = 2$$

b) $$\int_0^3 \int_{x^2}^9 \frac{x}{\sqrt{3x^2+y}}\,dy\,dx = \int_0^9 \int_0^{\sqrt{y}} \frac{x}{\sqrt{3x^2+y}}\,dx\,dy$$




$$= \int_0^9 \frac{1}{3}\sqrt{3x^2+y}\Big|_{x=0}^{\sqrt{y}} dy$$

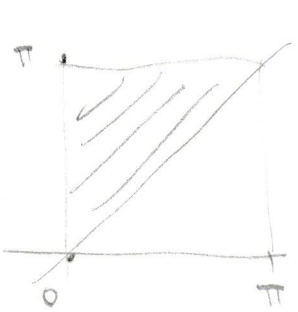
$$= \frac{1}{3}\int_0^9 \sqrt{4y} - \sqrt{y}\;dy$$

$$= \frac{2}{3}\frac{2}{3}\,y^{3/2}\Big|_{y=0}^{9} - \frac{1}{3}\frac{2}{3}\,y^{3/2}\Big|_{y=0}^{9} = 12 - 6 = 6.$$

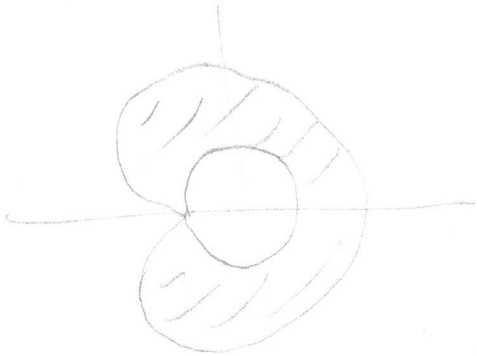
c) $$\int_0^\pi \int_x^\pi \frac{\cos y}{y}\,dy\,dx = \int_0^\pi \int_0^y \frac{\cos y}{y}\,dx\,dy = \int_0^\pi \frac{\cos y}{y}\,x\Big|_{x=0}^{y} dy = \int_0^\pi \cos y\,dy$$

$$= \sin y\Big|_0^\pi$$

$$= 0.$$



d)



$$\int_0^{2\pi}\int_1^{2+\cos\theta} \underline{\underline{r}}\, dr\, d\theta = \int_0^{2\pi} \frac{r^2}{2}\Bigg|_{r=1}^{r=2+\cos\theta} d\theta$$

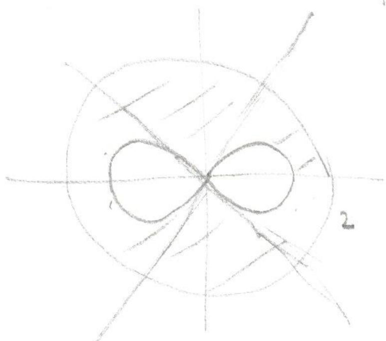
$$\| $$

$$\int_0^{2\pi} 2+\cos\theta + \frac{\cos\theta^2}{2} - \frac{1}{2}\, d\theta$$

$$= 4\pi + 0 + \frac{\pi}{2} - \pi = \underline{\frac{7\pi}{2}}$$

e)

Separate integral:





$$4\int_0^{\pi/4}\int_{\sqrt{\cos 2\theta}}^{2} \underline{\underline{r}}\, dr\, d\theta = 4\int_0^{\pi/4} \frac{r^2}{2}\Bigg|_{r=\sqrt{\cos 2\theta}}^{r=2} d\theta$$

$$= 4\int_0^{\pi/4} 2 - \frac{\cos 2\theta}{2}\, d\theta =$$

$$= 4\left(\frac{\pi}{2} - \frac{\sin 2\theta}{4}\Bigg|_{\theta=0}^{\pi/4}\right) = 2\pi - 1.$$

Other part $4\int_0^{\pi/4}\int_0^2 r\,dr\,d\theta = 4\int_0^{\pi/4} \frac{r^2}{2}\Big|_{r=0}^{r=2} d\theta = 4\int_0^{\pi/4} 2\,d\theta$

$= 2\pi.$

So: $\boxed{4\pi - 1}$

$\vec{\nabla} f = (2x, 2yz^2, 2y^2z)$ , so at $(2,-1,2)$ it is:

$(4, -8, 4)$. Thus tangent plane:

$$4x - 8y + 4z = 4\cdot 2 - 8(-1) + 4(2) = 24$$

$= \cancel{240}$

z-axis $= (x=0 \;\&\; y=0)$. In this case:

$\cancel{4z = 40}$ $\quad \cancel{\boxed{z = \frac{5}{2}}}$ $\qquad 4z = 24$ $\quad \boxed{z = 6}$

So intersection point is $\cancel{(0,0,4)}$. $(0,0,6)$.

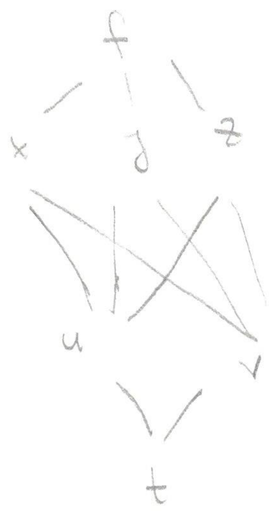
$\frac{\partial z}{\partial x} = e^{y-x} - xe^{y-x}$ $\qquad 0$

At $(1,1)$: $\qquad$ & $z(1,1) = 1$

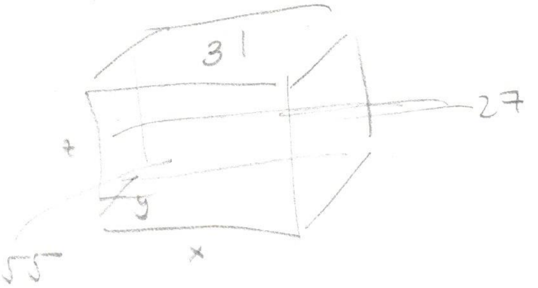
$\frac{\partial z}{\partial y} = xe^{y-x}$ $\qquad 1$

So $z = 1 + 1(y-1)$ is the approximation, and $(1,1,0.9)$

it gives $\underline{z = 0.9}$.




$$\boxed{\begin{aligned} \text{So} \quad \frac{\partial f}{\partial t} = &\frac{\partial f}{\partial x}\frac{\partial x}{\partial u}\frac{du}{dt} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u}\frac{du}{dt} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial u}\frac{du}{dt} \\ &+ \frac{\partial f}{\partial x}\frac{\partial x}{\partial v}\frac{dv}{dt} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v}\frac{dv}{dt} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial v}\frac{dv}{dt} \end{aligned}}$$




$$55xz + 31xy + \cancel{27}\,yz + 27yz + 27xz = f(x,y,z)$$

$$= 82xz + 54yz + 31xy$$

$$g(x,y,z) = xyz = 16\,000\,000$$

$$82z + 31y = \lambda yz$$

$$54z + 31x = \lambda xz$$

$$82x + 54y = \lambda xy$$

so

$$82xz + 31xy = \lambda xyz$$

$$54yz + 31xy = \lambda xyz$$

$$82xz + 54yz = \lambda xyz$$

So $82xz + \cancel{31xy} = 54yz + \cancel{31xy}$

$$82x = 54y \quad (z>0)$$

$$\frac{82x}{54} = y$$

So $\cancel{82xz} + 31xy = \cancel{82xz} + 54yz$

$$31x = 54z \quad \text{so}$$

$$\frac{31x}{54} = z.$$

So $x \cdot \frac{82x}{54} \cdot \frac{31x}{54} = 16\,000\,000$ so

$x \approx 263{,}78$

$y \approx 400{,}56$

$z \approx 151{,}43$


$g: \quad xy = 5000$

$f: \quad 2x + y$

$2 = \lambda y$ $\qquad$ clearly $\quad x \neq 0$, $\lambda \neq 0$

$1 = \lambda x$ $\qquad$ $\frac{2}{1} = \frac{y}{x}$

so $y = 2x$,

then $2x^2 = 5000$

$x^2 = 2500$

$\boxed{x = 50 \quad y = 100}$

So $2x + y = \boxed{200}$

2.

$20x + 30y = 600$ $\qquad g$

$10x^{0.6}y^{0.4}$ $\qquad f$

$10 \cdot 0.6\, x^{-0.4} y^{0.4} = \lambda\, 20$ $\qquad x, y \neq 0$ so $\lambda \neq 0$

$10 \cdot 0.4\, x^{0.6} y^{-0.6} = \lambda \cdot 30$

$\frac{0.6}{0.4} \frac{x^{-0.4}}{x^{0.6}} \frac{y^{0.4}}{y^{-0.6}} = \frac{20}{30}$

$= \frac{0.6}{0.4} \frac{y}{x} = \frac{20}{30}$

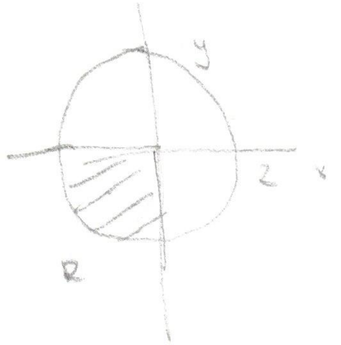
$\frac{18}{8} = \frac{x}{y} \Rightarrow \frac{9}{4}y = x$

$\underbrace{20\frac{9}{4}y}_{45y} + 30y = 600$

$\boxed{y = 8}$

$\boxed{x = 18}$

1)




$$\iint_R (x^2+y^2)\,dx\,dy$$

$$= \int_{\pi}^{3\pi/2}\int_0^2 r^2 \cdot \underline{\underline{r}}\; dr\,d\theta$$

$$= \int_{\pi}^{3\pi/2} \frac{r^4}{4}\Big|_0^2 \,d\theta = 4\int_{\pi}^{3\pi/2} d\theta = \boxed{2\pi}.$$

2) $f(x,y) = \ln(x^2+y^2)$

a) $\dfrac{\partial f}{\partial x} = \dfrac{2x}{x^2+y^2}$ $\qquad \dfrac{\partial f}{\partial y} = \dfrac{2y}{x^2+y^2}$ $\qquad$ at $(1,1)$: $\dfrac{\partial f}{\partial x} = 1 \quad \dfrac{\partial f}{\partial y} = 1$

$\vec{u} = |\vec{i} - \vec{j}| = \sqrt{2}$, so

$$D_{\vec{u}} f = (1,1)\cdot\frac{(1,-1)}{\sqrt{2}} = \underline{\underline{0}}.$$

b) $\vec{v}$ is in the direction of the gradient. $|\vec{i}+\vec{j}| = \sqrt{2}$.

$$D_{\vec{u}} f = (1,1)\cdot\frac{(1,1)}{\sqrt{2}} = \underline{\underline{\sqrt{2}}}$$

3)


$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r}$$

$$= 2\cos(2x-y)\,1 - \cos(2x-y)\,s$$

if $r = \pi$, $s = 0$ : $x = \pi$ and $y = 0$, so

$$\frac{\partial z}{\partial r} = 2\cos 2\pi - \cos 2\pi \cdot 0 = \boxed{2.}$$