Five Minutes of Logic
(Spring'00: Math 141) *

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Math 141, TTh 9-10:30 DRL A2

Contents

1 Proofs

Definition 1.1 By a *proof* of the assertion $A \to B$, we mean a finite list of statements

$$W_1, W_2, \ldots, W_n$$

where

(a) $W_1 = A$ (the *hypothesis*),

(b) $W_n = B$ (the *conclusion*),

(c) each of the statements W_i is

  (c1) either an *axiom*

  (c2) or a consequence of the previous statements: $W_1, W_2, \ldots, W_{i-1}$, according to a certain logical *rule of inference*.

2 Logical Connectives

The study of logic is concerned with the *truth* or *falseness* of statements.
We invoke the following logical connectives:

(i) negation $\neg$

*These are not a substitute for taking your own notes, but will be a help if the dog eats your notes.

(ii) conjunction &

(iii) disjunction $\vee$

(iv) implication $\to$

Definition 2.1 A statement H constructed from various substatements A, B, C, ..., and which is true no matter whether A, B, C, ... are true or false, is called a *tautology*.

2.1 De Morgan Dual Laws

Proposition 2.1

$$\neg(A \vee B) \quad = \quad (\neg A)\,\&\,(\neg B), \tag{1}$$
$$\neg(C\,\&\,D) \quad = \quad (\neg C) \vee (\neg D). \tag{2}$$

3 Quantifiers. Universal Quantifier. Existential Quantifier

Let $A(x)$ denote an assertion about x:
For each choice c of x (whatever value c of x we take), the assertion $A(c)$ is either true or false.

(a) "For every $x \in S$, $A(x)$ is true" is abbreviated as

$$\forall x \in S : A(x) \tag{3}$$

The statement $\forall x \in S : A(x)$ is *true* provided that $A(c)$ is true for every choice c of x in S.
For a finite S, say

$$S = \{c_1, c_2, .., c_m\},$$

the statement $\forall x \in S : A(x)$ means the same as

$$A(c_1) \& A(c_2) \& \cdots \& A(c_m).$$

(b) "For some $x \in S$, $A(x)$ is true" is abbreviated as

$$\exists x \in S : A(x) \tag{4}$$

The statement $\exists x \in S : A(x)$ is *true* provided that $A(c)$ is true for at least one choice c of x in S.
For a finite S, say

$$S = \{c_1, c_2, .., c_m\},$$

the statement $\exists x \in S : A(x)$ means the same as

$$A(c_1) \vee A(c_2) \vee \cdots \vee A(c_m).$$

Thus, the universal and existetial quantifiers are extensions of the connectives & and $\vee$, respectively, to deal with infinitely many assertions $A(c)$ about infinitely many *values* c.

Warning:

(a) "$\forall x \in S : A(x)$" means that $\forall x((x \in S) \to A(x))$,

(b) whereas "$\exists x \in S : A(x)$" means that $\exists x((x \in S) \& A(x))$.

3.1 De Morgan Dual Laws

Proposition 3.1

$$\neg\exists x\in S : A(x) \quad = \quad \forall x\in S : \neg A(x), \tag{5}$$
$$\neg\forall x\in T : B(x) \quad = \quad \exists x\in T : \neg B(x). \tag{6}$$

Definition 3.1 A sequence $x_1, x_2, \ldots, x_n, \ldots$ is a *Cauchy sequence* iff whatever $\varepsilon > 0$ we take, one can find a (non-negative) integer N such that for any positive k:

$$|x_{n+k} - x_n| < \varepsilon \ \text{ whenever } n > N.$$

Definition 3.1 incorporates *four* quantifying phrases: "whatever $\varepsilon > 0$", "one can find an integer N", "for any positive k", and "whenever $n > N$". In formal terms, it is abbreviated as:

$$\forall\varepsilon > 0\ \exists N\in\mathcal{N}\ \forall k > 0\ \forall n\geq N\ (|x_{n+k} - x_n| < \varepsilon). \tag{7}$$

What does it mean that the sequence $x_1, x_2, \ldots, x_n, \ldots$ is *not* a Cauchy sequence ?
Literally, it means that

$$\neg\forall\varepsilon > 0\ \exists N\in\mathcal{N}\ \forall k > 0\ \forall n\geq N\ (|x_{n+k} - x_n| < \varepsilon). \tag{8}$$

Indeed, the *duality principle* provides us with the following *positive* version of the *negative* (8):

$$\exists\varepsilon > 0\ \forall N\in\mathcal{N}\ \exists k > 0\ \exists n\geq N\ (|x_{n+k} - x_n| \geq \varepsilon). \tag{9}$$

3.2 Renaming Dummy Variables

Proposition 3.2 *The* quantified variable x *in the above statements (3) and (4) is a* bound*, or* "dummy"*, variable, and can be replaced (renamed) in all its occurrences by any other variable symbol:*

(1) *A statement of the form*

$$\forall x\in S : A(x),$$

means the same as

$$\forall y\in S : A(y),$$
$$\forall\varepsilon\in S : A(\varepsilon),$$
$$\forall\varepsilon'\in S : A(\varepsilon'),$$

etc.

(2) *A statement of the form*

$$\exists x\in S : A(x),$$

means the same as

$$\exists y\in S : A(y),$$
$$\exists\delta\in S : A(\delta),$$
$$\exists\delta_1\in S : A(\delta_1),$$

etc.

For instance,

$$\forall\varepsilon > 0\ \exists N\in\mathcal{N}\ \forall n\geq N\ (|x_n - a| < \varepsilon) \tag{10}$$

can be rewritten as

$$\forall\varepsilon' > 0\ \exists M\in\mathcal{N}\ \forall k\geq M\ (|x_k - a| < \varepsilon') \tag{11}$$

3.3 ∀∃ versus ∃∀

Proposition 3.3 *The order in which quantifiers appear affects the meaning of the statement.*

Proof. Compare:

$$\forall x \in \mathcal{R}\ \exists y \in \mathcal{R}\ ((x+y)=0)$$

and

$$\exists y \in \mathcal{R}\ \forall x \in \mathcal{R}\ ((x+y)=0).$$

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Proposition 3.4 *If an existential expression $\exists y \in S$ occurs in a given statement, then the choice of y generally depends on any variable that occurs* before *our y, but not on variables that occurs* after *y, (furthermore, in turn, our y affects those "*after *variables").*

(1) *In a statement of the form:*

$$\forall x \in S\ \exists y \in T : A(x,y), \tag{12}$$

the choice of y is allowed to depend *on the value of x;*
in other words, it means that there is a function f from S to T such that

$$\forall x \in S : A(x, f(x)), \tag{13}$$

(whatever c from S we take, $A(c, f(c))$ is true);

(2) *Whereas in the statement:*

$$\exists y \in T\ \forall x \in S : A(x,y), \tag{14}$$

the choice of y must be independent *of x.*

To emphasize that in statement (12) the choice of y *depends* on x, statement (12) may be rewritten as:

$$\forall x \in S\ \exists y_x \in T : A(x, y_x), \tag{15}$$

A "standardized" representation of statements within the rigid logical framework allows us to simplify and clarify the problems under consideration.

Definition 3.2 A sequence $x_1, x_2, \ldots, x_n, \ldots$ has a *limit* a iff whatever $\varepsilon > 0$ we take, one can find a positive integer N such that

$$|x_n - a| < \varepsilon \ \text{ whenever } n > N.$$

Along the above lines, keeping the information "who affects whom", we may say that

"the sequence $x_1, x_2, \ldots, x_n, \ldots$ has the *limit* a"

means that:

$$\forall \varepsilon > 0\ \exists N_\varepsilon \in \mathcal{N}\ \forall n \geq N_\varepsilon\ (|x_n - a| < \varepsilon) \tag{16}$$

Similarly, a sequence $y_1, y_2, \ldots, y_n, \ldots$ has a *limit* b iff

$$\forall \varepsilon > 0\ \exists N_\varepsilon \in \mathcal{N}\ \forall n \geq N_\varepsilon\ (|y_n - b| < \varepsilon) \tag{17}$$

But, it is extremely misleading to use one and the same symbols ε, N_ε in unrelated situations: there is no connection between N_ε in (16) and N_ε in (17). Based on Proposition 3.2, we can circumvent such a collision by rewriting (16) as:

$$\forall \varepsilon' > 0 \; \exists N_{\varepsilon'} \in \mathcal{N} \; \forall n \geq N_{\varepsilon'} \; (|x_n - a| < \varepsilon') \tag{18}$$

and by rewriting (17) as:

$$\forall \varepsilon'' > 0 \; \exists M_{\varepsilon''} \in \mathcal{N} \; \forall m \geq M_{\varepsilon''} \; (|y_m - b| < \varepsilon'') \tag{19}$$

It should be pointed out that we meet with the problem of "dummy" variables in all branches of mathematics: for instance,

$$\sum_{i=1}^{i=n} x_i = \sum_{j=1}^{j=n} x_j = \sum_{k=1}^{k=n} x_k = \dots$$

$$\int_a^b sin(2t+1)dt = \int_a^b sin(2\varphi+1)d\varphi = \int_a^b sin(2x+1)dx = \dots$$