Math 361

Basic facts from Math 360.

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Bounded Sets A subset S of the real numbers is bounded from below if there is some number m so that

$$m \leq x, \forall x \in S,$$

and bounded from above if there is a number M such that

$$M \geq x, \forall x \in S.$$

If a set is bounded from above and below, then we say it is bounded. If a set is bounded from below then we define $\inf S$ as the least upper bound of the set

$$\{m : \quad m \leq x, \; \forall x \in S\}.$$

If S is bounded from above we define $\sup S$ to be the greatest lower bound of the set

$$\{M : \quad x \leq M, \; \forall x \in S\}.$$

Sequences The most important idea covered in math 360 was the concept of a sequence. A sequence is a function from the positive integers (natural numbers $= \mathbb{N}$.) to some set. The simplest examples aroe sequences of real numbers, $\mathbb{R}$. Using standard functional notation we could denote the function as

$$x : \mathbb{N} \longrightarrow \mathbb{R},$$

then the n^{th} term of the sequence would be denoted $x(n)$. It is customary not to use functional notational but rather to denote the use subscripts, so the n^{th} term is denoted by x_n. Almost as important as the concept of a sequence is the concept of a subsequence. Given a sequence $\{x_n\}$, we define a subsequence by selecting a subset of $\{x_n\}$ and keeping them in the same order as they appear in $\{x_n\}$. In practice this amounts to defining a function from $\mathbb{N}$ to itself. We denote this function by n_j. It must have the following property:

$$n_j < n_{j+1}.$$

The j^{th} term of our subsequence is given by x_{n_j}. As an example, consider the sequence $x_n = (-1)^n n$; the mapping $n_j = 2j$ defines the subseqence $x_{n_j} = (-1)^{2j} 2j$.

Limits of Sequences A sequence of real numbers, $\{x_n\}$ has a limit if there is a number L such that given any $\epsilon > 0$ there exists a $N > 0$ such that

$$|x_n - L| < \epsilon \text{ whenever } n > N.$$

A sequence with a limit is called a convergent sequence. The limit, when it exists is unique. A sequence may itself fail to have limit but it may have a subsequence which does. In this case the sequence is said to have a convergent subsequence. For example $x_n = (-1)^n$ is not convergent but the subsequence defined by $n_j = 2j$ is.

Rules for Limits We have several rules for computing algebraic combinations of convergent sequences.

Algebraic Rules for Limits. *Suppose that $\{x_n\}, \{y_n\}$ are convergent sequences of real numbers then*

$$\lim_{n\to\infty} -x_n \text{ exists and equals } -\lim_{n\to\infty} x_n,$$
$$lim_{n\to\infty}(x_n + y_n) \text{ exists and equals } \lim_{n\to\infty} x_n + \lim_{n\to\infty} y_n,$$
$$lim_{n\to\infty}(x_n y_n) \text{ exists and equals } (\lim_{n\to\infty} x_n)(\lim_{n\to\infty} y_n),$$
$$lim_{n\to\infty}\frac{x_n}{y_n} \text{ exists, provided } \lim_{n\to\infty} y_n \neq 0, \text{ and equals } \frac{\lim_{n\to\infty} x_n}{\lim_{n\to\infty} y_n}.$$

In this theorem the non–trivial claim is that the limits exist, once this is clear it is easy to show what they must be.

Bolzano-Weierstrass A problem of fundamental importance is to decide whether or not a sequence has a limit. A sequence $\{x_n\}$ is bounded if there is a number M such that

$$|x_n| < M \text{ for all } n.$$

A sequence is non–increasing if

$$x_n \geq x_{n+1} \text{ for all } n,$$

and non–decreasing if

$$x_n \leq x_{n+1} \text{ for all } n.$$

The completeness axiom of the real numbers states

Completeness Axiom A bounded non–increasing or non–decreasing sequence has a limit.

If a bounded sequence is neither non–decreasing, nor non–increasing then the only general theorem about convergence is

Bolzano–Weierstrass Theorem. *A bounded sequence of real numbers always has a convergent subsequence.*

Note that this does not assert that any bounded sequence converges but only that any bounded sequence has some subsequence which converges.

In general, if $S \subset \mathbb{R}$ then the set of points which can be obtained as limits of sequences $\{x_n\} \subset S$ is called the set of accumulation points of S. A subset S is said to be dense in an interval I whenever I is a subset of the set of accumulation points of S. For example the rational numbers $\mathbb{Q}$ are dense in every interval. The following two lemmas are very useful

Lemma 1. *If x_n, y_n, z_n are sequences of real numbers such that*

$$x_n \leq y_n \leq z_n$$

and x_n and z_n are convergent with

$$L = \lim_{n\to\infty} x_n = \lim_{n\to\infty} z_n$$

then y_n converges with

$$\lim_{n\to\infty} y_n = L.$$

Lemma 2. *If $x_n \geq 0$ is convergent then*

$$\lim_{n\to\infty} x_n \geq 0.$$

Cauchy Sequences In the above discussion of limits we always assumed that the limiting value is known in advance. There is a criterion due to Cauchy which implies that a given sequence has a limit which makes no reference to its value.

Cauchy Criterion for Sequences. *If $\{x_n\}$ is a sequence of real numbers such that given $\epsilon > 0$ there exists an N for which*

$$|x_n - x_m| < \epsilon \text{ whenever both } n \text{ and } m \text{ are greater than } N,$$

then the sequence is convergent.

Series A series is the sum of a sequence. This is usually denoted by

$$\sum_{n=1}^{\infty} x_n.$$

A *series* converges if the *sequence* of partial sums

$$s_k = \sum_{n=1}^{k} x_n,$$

converges. In this case the sum of the series is defined by:

$$\sum_{n=1}^{\infty} x_n = \lim_{k\to\infty} s_k.$$

If a series does not converge then it diverges. A series converges *absolutely* if the sum of the absolute values

$$\sum_{n=1}^{\infty} |x_n|$$

converges. The following theorem describes the elementary properties of series.

Theorem on Series. *Suppose that x_n, y_n are sequences. Suppose that $\sum x_n, \sum y_n$ converge then*

$$\sum (x_n + y_n) \text{ converges and } \sum (x_n + y_n) = \sum x_n + \sum y_n,$$
$$\text{If } a \in \mathbb{R} \sum a x_n = a \sum x_n,$$
$$\text{If } x_n \geq 0 \text{ for all } n, \text{ then } \sum x_n \geq 0.$$

Convergence Tests There are many criteria that are used to determine if a given series converges. The most important is the comparison test

Comparison Test. *Suppose that x_n, y_n are sequences such that $|x_n| \leq y_n$ if $\sum y_n$ converges then so does $\sum x_n$. If $0 \leq y_n \leq x_n$ and $\sum y_n$ diverges then so does $\sum x_n$.*

To apply this test we need to have examples of series which we know are convergent or divergent. The simplest case is a geometric series. This is because we have a formula for the partial sums:

$$\sum_{n=0}^{k} a^k = \frac{a^{k+1} - 1}{a - 1}.$$

From this formula we immediately conclude

Convergence of Geometric Series. *A geometric converges if and only if $|a| < 1$.*

The root and ratio tests are really special cases of the comparison test where the series are comparable to geometric series.

Ratio Test. *If x_n is a sequence with*

$$\limsup_{n\to\infty}\left|\frac{x_{n+1}}{x_n}\right| = \alpha$$

then the series

$$\sum_{n=1}^{\infty} x_n \begin{cases} \text{converges if} & \alpha < 1 \\ \text{diverges if} & \alpha > 1. \end{cases}$$

The test gives not information if $\alpha = 1$.

We also have

Root Test. *If x_n is a sequence with*

$$\limsup_{n\to\infty} |x_n|^{\frac{1}{n}} = \alpha$$

then the series

$$\sum_{n=1}^{\infty} x_n \begin{cases} \text{converges if} & \alpha < 1 \\ \text{diverges if} & \alpha > 1. \end{cases}$$

The test gives not information if $\alpha = 1$.

If $\alpha < 1$ in the ratio or root tests then the series converge absolutely. Another test is obtained by comparing a series to an integral

A Convergence Test. *If x_n is a monotone decreasing, positive sequence then*

$$\sum_{n=1}^{\infty} x_n \text{ converges if and only if } \sum_{k=1}^{\infty} 2^k x_{2^k}.$$

Using this test we can easily show that the sum,

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

converges if and only if $p > 1$. A final test which is sometimes for showing that a series with terms that alternate in sign converges is:

Alternating Series Test. *Suppose that x_n is a sequence such that the sign alternates, the* $\lim_{n\to\infty} x_n = 0$ *and* $|x_{n+1}| \leq |x_n|$ *then*

$$\sum_{n=1}^{\infty} x_n$$

converges.

Note that this test requires that the signs alternate, the absolute value of the sequence is monotonely decreasing and the sequence tends to zero. If any of these conditions are not met the series may fail to converge. This test is a consequence of a very useful formula, the partial summation formula:

$$\sum_{n=1}^{m} a_n B_n = -\sum_{n=1}^{m} b_n(a_{n+1} - a_n) + a_{m+1} B_m,$$

$$\text{where } B_n = \sum_{i=1}^{n} b_i.$$

Limits of Functions The next thing to consider is the behavior of functions defined on intervals in $\mathbb{R}$. Suppose that $f(x)$ is defined for $x \in (a, c) \cup (c, b)$. This is called a punctured neighborhood of c. We say that o$f(x)$ has a limit, L as x approaches c if given $\epsilon > 0$ there exists $\delta > 0$ such that

$$|f(x) - L| < \epsilon \text{ provided } 0 < |x - c| < \delta$$

and we write

$$\lim_{x \to c} f(x) = L.$$

Note that in this definition nothing is said about the value of $f(x)$ at $x = c$. This has no bearing at all on whether the limit exists. If $f(c)$ is defined and we have that

$$\lim_{x \to c} f(x) = f(c)$$

then we say that $f(x)$ is continuous at $x = c$. If $f(x)$ is continuous for all $x \in (a, b)$ then we say that $f(x)$ is continuous on (a, b). In addition to the ordinary limit we also define one sided limits. If $f(x)$ is defined in (a, b) and there exists an L such that given $\epsilon > 0$ there exists δ such that

$$|f(x) - L| < \epsilon \text{ provided } 0 < x - a < \delta \text{ then } \lim_{x \to a^+} f(x) = L.$$

If instead

$$|f(x) - L| < \epsilon \text{ provided } 0 < b - x < \delta \text{ then } \lim_{x \to b^-} f(x) = L.$$

The rules for dealing with limits of functions are very similar to the rules for handling limits of sequences

Algebraic Rules for Limits of Functions. *Suppose that $f(x), g(x)$ are defined in a punctured neighborhood of c and that*

$$\lim_{x \to c} f(x) = L, \quad \lim_{x \to c} g(x) = M.$$

Then

$$\begin{aligned}
\lim_{x \to c}(-f(x)) &\text{ exists and equals } -L,\\
\lim_{x \to c}(f(x) + g(x)) &\text{ exists and equals } L + M,\\
\lim_{x \to c}(f(x)g(x)) &\text{ exists and equals } LM,\\
\lim_{x \to c} \frac{f(x)}{g(x)} &\text{ exists, provided } M \neq 0 \text{ and equals } \frac{L}{M}.
\end{aligned}$$

From this we deduce the following results about continuous functions

Algebraic Rules for Continuous Functions. *If $f(x), g(x)$ are continuous at $x = c$ then so are $-f(x)$, $f(x) + g(x)$, $f(x)g(x)$. If $g(c) \neq 0$ then $f(x)/g(x)$ is also continuous at $x = c$.*

For functions we have one further operation which is very important, composition.

Continuity of Compositions. *Suppose that $f(x), g(y)$ are two functions such that $f(x)$ is continuous at $x = c$ and $g(y)$ is continuous at $y = f(c)$ then the composite $g \circ f(x)$ is continuous at $x = c$.*

Uniform Continuity A function defined on an interval $[a, b]$ is said to be uniformly continuous if given $\epsilon > 0$ there exists δ such that

$$|f(x) - f(y)| < \epsilon, \forall x, y \in [a, b] \text{ with } |x - y| < \delta.$$

The basic proposition is

Proposition. *A continuous function on a closed bounded interval is uniformly continuous.*

Using similar arguments we can also prove

Max–Min theorem for Continuous Functions. *If $f(x)$ is continuous on a closed bounded interval, $[a,b]$ then there exists $x_1 \in [a,b]$ and $x_2 \in [a,b]$ which satisfy*

$$f(x_1) = \sup_{x\in[a,b]} f(x), \quad f(x_2) = \inf_{x\in[a,b]} f(x).$$

As a final result on continuous functions we have Intermediate Value Theorem

Intermediate Value Theorem. *Suppose that $f(x)$ is continuous on $[a,b]$ and $f(a) < f(b)$ then given $y \in (f(a), f(b))$ there exists $c \in (a,b)$ such that $f(c) = y$.*

Differentiability A function defined in a neighborhood of a point c is said to be differentiable at c if the function

$$g(x) = \frac{f(x) - f(c)}{x - c},$$

defined in a deleted neighborhood of c has a limit as $x \to c$. This limit is called the derivative of f at c; we denote it by $f'(c)$. A function which is differentiable at every point of an interval is said to be differentiable in the interval. If the derivative is itself continuous then the function is said to be continuously differentiable. As with continuous functions we have algebraic rules for differentiation.

Rules for Differentiation. *Suppose that $f(x), g(x)$ are differentiable at $x = c$ then so are $-f(x), (f(x) + g(x)), f(x)g(x)$. If $g(c) \neq 0$ then so is $f(x)/g(x)$. The derivatives are given by*

$$\begin{aligned}
(-f)'(c) &= -(f'(c)), \\
(f+g)'(c) & f'(c) + g'(c), \\
(fg)'(c) &= f'(c)g(c) + f(c)g'(c), \\
\left(\frac{f}{g}\right)'(c) &= \frac{f'(c)g(c) - f(c)g'(c)}{g(c)^2}.
\end{aligned}$$

In addition we can also differentiate a composition

The Chain Rule. *If $f(x)$ is differentiable at $x = c$ and $g(y)$ is differentiable at $y = f(c)$ then $g \circ f(x)$ is differentiable at $x = c$; the derivative is*

$$g \circ f'(c) = g'(f(c))f'(c).$$

Little o and big O It is often useful to be able to compare the size of two functions $f(x), g(x)$ near a point $x = c$ without being too specific. When we write

$$f(x) = o(g(x)) \text{ near to } x = c$$

it means that

$$\lim_{x\to c} \frac{|f(x)|}{g(x)} = 0.$$

If we write the

$$f(x) = O(g(x)) \text{ near to } x = c$$

this means that we can find an M and an $\epsilon > 0$ such that

$$|f(x)| < Mg(x) \text{ provided } |x - c| < \epsilon \text{ i.e. } \limsup_{x\to c} \frac{|f(x)}{g(x)} < \infty.$$

For example a function $f(x)$ is differentiable at $x = c$ if and only if there exists a number L for which

$$f(x) = f(c) + L(x - c) + o(|x - c|).$$

Of course $L = f'(c)$.

Higher Order Derivatives If the derivative of function $f'(x)$ happens itself to be differentiable then we say that $f(x)$ is twice differentiable. The second derivative is denoted by $f''(x)$. Inductively if the k^{th} derivative happens to be differentiable then we say that f is $k + 1$–times differentiable. We denote the k^{th} derivative by $f^{[k]}(x)$.

Taylor's Theorem For a function which has n derivatives we can find a polynomial which agrees with $f(x)$ to order $n - 1$ at a point.

Taylor's Theorem. *Suppose that $f(x)$ has n derivatives at a point c then*

$$f(x) - \sum_{j=0}^{n-1} \frac{f^{[j]}(c)(x - c)^j}{j!} + R_n(x)$$

where

$$R_n(x) = O(|x - c|^n).$$

There are many different formulæ for the error term $R_n(x)$. One such expression is

$$R_n(x) = \frac{f^{[n]}(\alpha)}{n!}(x - c)^n \text{ where } \begin{cases} \alpha \in & (x, c) \text{ if } x < c \\ \alpha \in & (c, x) \text{ if } x > c \end{cases}$$

An important special case of Taylor's Theorem is the mean value theorem

Mean Value Theorem. *Suppose that $f(x)$ is continuous on $[a, b]$ and differentiable on (a, b) then there exists a $c \in (a, b)$ such that*

$$f(b) - f(a) = f'(c)(b - a).$$

Open and Closed sets A subset S of $\mathbb{R}$ is said to be open if for each $x \in S$ there is an $\epsilon > 0$ such that

$$(x - \epsilon, x + \epsilon) \subset S.$$

A set S is said to be closed if its complement $S^c = \mathbb{R} \setminus S$ is open.

Properties of open sets. *If $\{S_\alpha : \quad \alpha \in \mathcal{A}\}$ is a collection of open set then*

$$\bigcup_{\alpha \in \mathcal{A}} S_\alpha \textit{ is open }.$$

If $\mathcal{A}$ is a **finite** *set then*

$$\bigcap_{\alpha \in \mathcal{A}} S_\alpha \textit{ is open }.$$

Using De Morgan's laws

$$(A \cap B)^c = A^c \cup B^c, \quad (A \cup B)^c = A^c \cap B^c$$

one easily derives

Properties of closed sets. *If $\{S_\alpha : \quad \alpha \in \mathcal{A}\}$ is a collection of closed set then*

$$\bigcap_{\alpha \in \mathcal{A}} S_\alpha \text{ is open .}$$

If $\mathcal{A}$ is a **finite** *set then*

$$\bigcup_{\alpha \in \mathcal{A}} S_\alpha \text{ is open .}$$

If $S \subset \mathbb{R}$ then we define the set of limit points of S by

$$S' = \{x : \quad \text{for any } \epsilon > 0 (x-\epsilon, x+\epsilon) \cap S \setminus \{x\} \neq \emptyset\}.$$

Characterization of closed sets by limit points. *A subset $S \subset \mathbb{R}$ is closed if and only if $S' \subset S$.*

For an arbitrary subset we define the closure of S by

$$\operatorname{cl} S = \overline{S} = S \cup S'.$$

This is always a closed set. We also define the interior of S by

$$\operatorname{int} A = \bigcup_{A \subset S, A \text{ open}} A.$$

Compact sets A set $S \subset \mathbb{R}$ is compact if the following property holds

Every sequence $\{x_n\} \subset S$ has a limit point in S.

Heine–Borel Theorem. *A set S is compact if and only if it is closed and bounded.*

A collection of sets $\mathcal{B} = \{B_\alpha : \quad \alpha \in \mathcal{A}\}$ define a cover of a set, S provided:

$$S \subset \bigcup_{\alpha \in \mathcal{A}} B_\alpha.$$

If each B_α is open set then we say that $\mathcal{B}$ defines an open cover of S. If there is a finite subset $\{\alpha_1, \ldots, \alpha_m\} \subset \mathcal{A}$ such that

$$S \subset B_{\alpha_1} \cup \cdots \cup B_{\alpha_m}$$

then we say that $\mathcal{B}$ contains a finite subcover. We have the following alternative characterization of compact sets:

Open cover definition of compactness. *A subset $S \subset \mathbb{R}$ is compact if and only if every open cover of S contains a finite subcover.*

One of the reasons compactness is an important property is the following:

Continuous functions on compact sets. *If f is a continuous function defined on the compact set, S then $f(S)$ is compact.*

This result has a very important corollary

Corollary. *A continuous function assumes its minimum and maximum values on a compact set.*

Connected sets A subset S of $\mathbb{R}$ is said to be disconnected if there exists two open sets A, B such that

$$\begin{gathered} A \cap B = \emptyset, \\ S \subset A \cup B, \\ A \cap S \neq \emptyset \text{ and } B \cap S \neq \emptyset. \end{gathered}$$

The only connected subsets of $\mathbb{R}$ are rays and intervals:

$$(a, b), (a, b], [a, b), [a, b].$$

Let $S \subset \mathbb{R}$ and suppose that $A \subset S$ has the following properties

$$\begin{gathered} A \text{ is connected,} \\ \text{If } B \text{ is a connected subset of } S \text{ then either } B \subset A \text{ or } A \cap B = \emptyset. \end{gathered}$$

We say that A is a connected component of S. The connected components of a set are its largest connected subsets. One can show that the connected components of an open sets are all of the form (a, b) and establish the following

Structure of open sets. *If S is an open subset of $\mathbb{R}$ then there is a countable collection of disjoint intervals $\{(a_j, b_j)\}$ such that*

$$S = \bigcup_{j=1}^{\infty} (a_j, b_j).$$

These intervals are the connected components of S.

One of the reasons connectedness is an important property is the following:

Continuous functions on connected sets. *If f is a continuous function defined on the connected set, S then $f(S)$ is connected.*

Power Series A very special type of infinite series is a power series:

$$\sum_{n=0}^{\infty} a_n x^n.$$

We define the number

$$\rho = \limsup_{n \to \infty} |a_n|^{\frac{1}{n}}.$$

Convergence for power series. *If $\rho = \infty$ then the power series diverges for all $x \neq 0$. If $\rho < \infty$ then the power series converges absolutely for $x \in (\frac{-1}{\rho}, \frac{1}{\rho})$ and diverges for $\rho|x| > 1$.*

In fact one can show that in case $\rho < \infty$ then the power series is an infinitely differentiable function of $x \in (\frac{-1}{\rho}, \frac{1}{\rho})$. Many other convergence criteria for power series can be obtained from the convergence criteria for numerical series given above.