Some solutions to HW 11

Problem 1) Suppose $\{x_i\}_{i\in I}$ is a net (in $\mathbb{R}$) which is bounded from above and satisfies $\forall i,j\in I: i\leq j \Rightarrow x_i \leq x_j$.

Claim: Then it converges to $\sup\{x_i \mid i\in I\}$.

Proof: We need to show that (denote $a=\sup\{x_i \mid i\in I\}$)

$$\forall \varepsilon>0\ \exists i_\varepsilon\in I: \forall j\geq i_\varepsilon : (a-\varepsilon < x_j < a+\varepsilon).$$

Let $\varepsilon>0$. By the definition of supremum, there exists $i_\varepsilon\in I$ s.t. $a-\varepsilon < x_{i_\varepsilon}$. Now, if $j\geq i_\varepsilon$ we have

$$a-\varepsilon < x_{i_\varepsilon} \leq x_j \leq a < a+\varepsilon$$

(because a upper bound)

So $a-\varepsilon < x_j < a+\varepsilon$ as we wanted. □

Problem 2) Want to prove: $\{x_i\}_{i\in I}$ converges $\Leftrightarrow \limsup x_i = \liminf x_i$.

"$\Rightarrow$": Say $x_i \to a \in\mathbb{R}$. It is enough to prove that

$$\limsup x_i \leq a \leq \liminf x_i .$$

Let $\varepsilon>0$. Then $\exists i_\varepsilon\in I$ such that

$$\forall j\geq i_\varepsilon: \quad a-\varepsilon < x_j < a+\varepsilon$$

$a\leq \liminf x_i$ $\qquad$ $a\geq \limsup x_i$

Thus $\limsup x_i = \liminf x_i = a = \lim x_i$.

"$\Leftarrow$": Now, assume that $\limsup x_i \leq a \leq \liminf x_i$.

Let $\varepsilon>0$, and Pick $\begin{cases} i_1\in I: & \forall j\geq i_1\ (a-\varepsilon < x_j) \\ i_2\in I: & \forall j\geq i_2\ (x_j < a+\varepsilon)\end{cases}$

As I is a directed set, $\exists i_\varepsilon$ s.t. $i_\varepsilon\geq i_1$ and $i_\varepsilon \geq i_2$.

Now for $j\geq i_\varepsilon$ we have $a-\varepsilon < x_j < a+\varepsilon$,

so $x_i\to a$. □

Pg 210 #1) If $f:[a,b]\to\mathbb{R}$ is $f(x)=1$, then for any partition $\underset{\substack{\shortparallel \\ a}}{x_0} < x_1 < \dots < \underset{\substack{\shortparallel \\ b}}{x_n}$, and any choice $c_i\in[x_{i-1},x_i]$, the Riemann sum is

$$\sum_{i=1}^{n} f(c_i)\cdot(x_i-x_{i-1}) = \sum_{i=1}^{n}(x_i-x_{i-1}) = x_n-x_0 = b-a.$$

Thus $\int_a^b f(x)\,dx = b-a$.

Pg 210 #2) Say $\underset{\substack{\shortparallel \\ 0}}{x_0} < \dots < \underset{\substack{\shortparallel \\ 3}}{x_n}$ is a partition of $[0,3]$, and $f(x)=x+5$.

As $f(x)$ is increasing, the upper sum is

$$U(P,f) = \sum_{i=1}^{n}(x_i-x_{i-1})\cdot f(x_i)$$

and the lower sum $L(P,f)=\sum_{i=1}^{n}(x_i-x_{i-1})f(x_{i-1})$.

If $\delta=\max\{x_i-x_{i-1} \mid i=1,\dots,n\}$ then the difference

$$U(P,f)-L(P,f) = \sum_{i=1}^{n}(x_i-x_{i-1})\big(f(x_i)-f(x_{i-1})\big) =$$

$$= \sum_{i=1}^{n}(x_i-x_{i-1})\cdot(x_i-x_{i-1}) \le \sum_{i=1}^{n}\delta(x_i-x_{i-1}) = 3\delta$$

As $\delta\to 0$, the difference goes to $0 \Rightarrow f$ integrable.

Use the partition $x_0<\dots<x_n$ where $x_i=\frac{3i}{n}$ for $i=0,\dots,n$.

Then the upper sum is

$$\sum_{i=1}^{n}\left(\frac{3i}{n}+5\right)\cdot\frac{3}{n} = \left(\frac{3}{n}\right)^2\left(\sum_{i=1}^{n} i\right) + \frac{3}{n}\cdot\left(\sum_{i=1}^{n}5\right) = \frac{9}{n^2}\cdot\frac{n(n+1)}{2} + \frac{15n}{n}$$

$$\to \frac{9}{2}+15 \quad \text{as } n\to\infty$$

$$\Rightarrow \int_0^3 (x+5)\,dx = \underline{\underline{\frac{9}{2}+15}} = \frac{39}{2}.$$

#8) We know that $F(y)-F(x)=\int_x^y f(t)\,dt$, so

$$|F(y)-F(x)| \le \int_x^y |f(t)|\,dt \le \int_x^y M\,dt = M|y-x|$$

$\Rightarrow F$ is M-Lipschitz $\Rightarrow F$ continuous.

Pg 234 #28) Yes, f is bounded: Pick $\delta > 0$ such that

$$\forall x,y \in (0,1): \ |f(x)-f(y)| < 1.$$

Now pick $n \in \mathbb{N}$ such that $\frac{1}{n} < \delta$, and

set $M = \max\{|f(\frac{i}{n})| \mid i = 1, \dots, n-1\} + 1$.

If $x \in (0,1)$, there exists $i \in \{1, \dots, n-1\}$ with

$$|x - \tfrac{i}{n}| \le \tfrac{1}{n} < \delta, \text{ so}$$

$$|f(x) - f(\tfrac{i}{n})| < 1 \Rightarrow |f(x)| \le M.$$

Thus f is bounded.

#42)

d) Obvious: $\int_1^1 1/t \, dt = 0$.

c) Follows from FTC.

a) Follows from c): $F'(x) = 1/x > 0 \Rightarrow F$ increasing

b) $$L(xy) = \int_1^x \frac{1}{t} dt + \underbrace{\int_x^{xy} \frac{1}{t} dt}_{\substack{\text{sub } t = xu \\ dt = x\,du}} = \int_1^x \frac{1}{t} dt + \int_{u=1}^{y} \underbrace{\frac{1}{xu} \cdot x\,du}_{\frac{1}{u} du} = L(x) + L(y)$$

e) $L(x) = \log(x)$ (natural log)

#45) Denote
$$m = \min\{g(x) \mid x \in [a,b]\} = g(x_1) \text{ for some } x_1 \in [a,b]$$
$$M = \max\{g(x) \mid x \in [a,b]\} = g(x_2) \text{ for some } x_2 \in [a,b].$$

Then $m \le g(x) \le M \quad \forall x \in [a,b]$

$$\Rightarrow m f(x) \le g(x) f(x) \le M f(x) \quad \text{as } f(x) \ge 0$$

$$\Rightarrow m \int_a^b f(x) dx \le \int_a^b g(x) f(x) dx \le M \int_a^b f(x) dx.$$

$\Rightarrow$

$$g(x_1) = m \le \frac{\int_a^b g(x) f(x) dx}{\int_a^b f(x) dx} \le M = g(x_2)$$

$\Rightarrow$ IVT: $\exists x_0$ between x_1, x_2 with

$$g(x_0) = \frac{\int_a^b g(x) f(x) dx}{\int_a^b f(x) dx} \quad \square$$