Some solutions to Homework 7

Problem 1) Suppose T is totally bounded in (M, d_1). We want to show that T is also totally bounded in (M, d_2).
For this, let $\varepsilon > 0$. We know that there is a constant $C > 0$ such that $d_2(x,y) \le C \cdot d_1(x,y)$ for all $x, y \in M$.
Now by our assumption, we can cover T by $\frac{\varepsilon}{C}$-disks: suppose $x_1, \dots, x_n \in T$ satisfy $T \subseteq D_1(x_1, \frac{\varepsilon}{C}) \cup \dots \cup D_1(x_n, \frac{\varepsilon}{C})$,
where $D_1(x_i, \frac{\varepsilon}{C}) = \{ y \in M \mid d_1(y, x_i) < \frac{\varepsilon}{C} \}$ is the sphere in (M, d_1).
Note that ~~if~~ $y \in D_1(x_i, \frac{\varepsilon}{C})$, then $d_2(y, x_i) \le C d_1(x_i, y) < C \cdot \frac{\varepsilon}{C} = \varepsilon$, so $y \in D_2(x_i, \varepsilon)$. Thus we get inclusion

$$D_2(x_1, \varepsilon) \cup \dots \cup D_2(x_n, \varepsilon) \supseteq D_1(x_1, \tfrac{\varepsilon}{C}) \cup \dots \cup D_1(x_n, \tfrac{\varepsilon}{C}) \supseteq T,$$

so the disks $D_2(x_i, \varepsilon)$ cover T. Since $\varepsilon > 0$ was arbitrary, we conclude that T is totally bounded in (M, d_2).

Problem 2) Suppose $T \subseteq M$ is totally bounded. We want to prove that its closure $\overline{T}$ is totally bounded. Let $\varepsilon > 0$.
Pick a covering of T by $\frac{\varepsilon}{2}$-disks:

$$T \subseteq D(x_1, \tfrac{\varepsilon}{2}) \cup \dots \cup D(x_n, \tfrac{\varepsilon}{2}) \quad \text{for some } x_i \in T.$$

Now I claim that $\overline{T} \subseteq D(x_1, \varepsilon) \cup \dots \cup D(x_n, \varepsilon)$.
If $y \in \overline{T}$, there is an $x \in T$ with $d(y, x) < \frac{\varepsilon}{2}$.
Then $x \in D(x_i, \frac{\varepsilon}{2})$ for some i, so $d(x, x_i) < \frac{\varepsilon}{2}$. By triangle ineq. we get $d(y, x_i) < \varepsilon$ so $y \in D(x_i, \varepsilon)$. As $y \in \overline{T}$ was arbitrary, we have $\overline{T} \subseteq D(x_1, \varepsilon) \cup \dots \cup D(x_n, \varepsilon)$ and $\overline{T}$ is totally bounded.

Pg 125, #2) Assume that every bounded sequence in M has a convergent sub-sequence.
We want to prove that every Cauchy sequence in M converges.
Let (x_n) be a Cauchy sequence in M. Every Cauchy sequence is bounded, so by assumption (x_n) has a convergent subsequence, say $x_{n_k} \to x \in M$ as $k \to \infty$.
Now we want to prove that in fact $x_n \to x$ as $n \to \infty$.
Let $\varepsilon > 0$. As (x_n) is Cauchy, there is N s.t. for $n, m \geq N$ we have $d(x_n, x_m) < \frac{\varepsilon}{2}$.
Since $\lim_{k\to\infty} x_{n_k} = x$, there is some $n_k > N$ such that $d(x_{n_k}, x) < \frac{\varepsilon}{2}$.
Now for all $n \geq N$ we have $d(x_n, x) \leq d(x_n, x_{n_k}) + d(x_{n_k}, x) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$,
so we get $\lim_{n\to\infty} x_n = x$, and M is complete.

Pg 125, #5

Let (x_n) be a Cauchy sequence in M. As (x_n) is bounded, by assumption it has a cluster point $x \in M$. We want to show that $\lim_{n\to\infty} x_n = x$. Let $\varepsilon > 0$. Take N such that for $n, m \geq N$ we have $d(x_n, x_m) < \frac{\varepsilon}{2}$. Since x is a cluster point of (x_n), there is a $k \geq N$ with $d(x_k, x) < \frac{\varepsilon}{2}$. But now for any $n \geq N$ we have $d(x_n, x) \leq d(x_n, x_k) + d(x_k, x) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$.
Thus $\lim_{n\to\infty} x_n = x$ and M is complete.

Pg 145, #13

"$\supseteq$": Clearly $A \subseteq cl(A)$. If $x \in bd(A)$ and $\varepsilon > 0$, then $D(x, \varepsilon) \cap A \neq \emptyset$, so $x \in cl(A)$. Thus $A \cup bd(A) \subseteq cl(A)$.

"$\subseteq$": Let $x \in cl(A)$. 1) If $\exists \varepsilon > 0 : (D(x, \varepsilon) \subseteq A)$, then $x \in int(A) \subseteq A$.
2) If $\nexists \varepsilon > 0 : (D(x, \varepsilon) \subseteq A)$, then $D(x, \varepsilon) \cap A^c \neq \emptyset$, $\forall \varepsilon > 0$.
OTOH, $D(x, \varepsilon) \cap A \neq \emptyset$ as $x \in cl(A)$, so by def. $x \in bd(A)$.
Thus $x \in A \cup bd(A)$ in both cases, so $cl(A) \subseteq A \cup bd(A)$.

Pg 145 # 14) In all of these, we will use the following fact:
if $A \subseteq B \subseteq M$, then $cl(A) \subseteq cl(B)$.
($cl(B)$ is closed set and it contains A, so it must contain $cl(A)$).

a) Note that $cl(A)$ is already a closed set, so its closure is itself. Thus $cl(cl(A)) = cl(A)$.

b) "$\supseteq$": As $A \subseteq A \cup B$ and $B \subseteq A \cup B$, we have $cl(A) \subseteq cl(A \cup B)$ and $cl(B) \subseteq cl(A \cup B)$.
Thus $cl(A) \cup cl(B) \subseteq cl(A \cup B)$.

"$\subseteq$" As $A \subseteq cl(A)$, $B \subseteq cl(B)$, we have $A \cup B \subseteq cl(A) \cup cl(B)$.
Since $cl(A) \cup cl(B)$ is a closed set containing $A \cup B$, it contains $cl(A \cup B)$, so $cl(A) \cup cl(B) \supseteq cl(A \cup B)$.

c) As $A \cap B \subseteq A$, and $A \cap B \subseteq B$, we have $cl(A \cap B) \subseteq cl(A)$ and $cl(A \cap B) \subseteq cl(B)$.
Thus $cl(A \cap B) \subseteq cl(A) \cap cl(B)$.

Pg 145 #17)

$\sum_{m=1}^{\infty} x_m$ converges absolutely

$\Rightarrow \sum_{m=1}^{\infty} |x_m| < \infty$

$\Rightarrow \sum_{m=1}^{\infty} |x_m \sin(m)| \leq \sum_{m=1}^{\infty} |x_m| < \infty$

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Pg 146 #26) Denote $f(x) = 1 + \frac{1}{1+x}$ for $x \in [1,2]$. Note that for $x \in [1,2]$,

$$1 \leq f(x) \leq 1 + \frac{1}{1+1} \leq 2 \text{ . so } f(x) \in [1,2].$$

As $a_0 = 1$ and $f(a_{n-1}) = a_n$, we see that $a_n \in [1,2]$ for all $n \geq 0$.

If $x, y \in [1,2]$, we have $f(x) - f(y) = \frac{1}{1+x} - \frac{1}{1+y} = \frac{y-x}{(1+x)(1+y)}$, so

$$|f(x) - f(y)| \leq \frac{1}{4}|x-y|.$$

This means that $|a_{n+1} - a_n| \leq \left(\frac{1}{4}\right)^n \Rightarrow (a_n)$ Cauchy $\Rightarrow$ Converges.
limit sat. $x = 1 + \frac{1}{1+x} \Rightarrow \boxed{x = \sqrt{2}}$

Pg 155 #1) Definitions:

A is sequentially compact if every sequence $x_n \in A$ has $\underbrace{\ }_{\text{subseq. with}}$ a limit $x \in A$

Every infinite ~~subsequence~~ subset ~~of~~ $S \subseteq A$ has an accumulation point in A: i.e. $x \in A$ that satisfies $\forall \varepsilon > 0$ there are infinitely many $y \in S$ with $d(x,y) < \varepsilon$.

"$\Leftarrow$": Suppose $x_n \in A$ is a sequence in A.
Define $S = \{x_n \mid n \in \mathbb{N}\} \subseteq A$. Two cases:

1) If S is finite, then there are (x_{n_k}) subsequence of (x_n), which is constant: all x_{n_k} are the same. Then
$$\lim_{k\to\infty} x_{n_k} \in A.$$

2) If S is infinite, then by assumption it has an accumulation point $x \in A$. We can then define a subsequence (x_{n_k}) recursively:
$$\begin{cases} n_1 = 1 \\ n_{k+1} = \min\left\{ n \in \mathbb{N} \;\middle|\; n > n_k \text{ and } d(x_n, x) < \frac{1}{k+1} \right\} \end{cases}$$
(Note that min. of an infinite subset of $\mathbb{N}$ always exists.)

Now $x_{n_k} \in A$ and $d(x_{n_k}, x) < \frac{1}{k}$ so $\lim\limits_{k\to\infty} x_{n_k} = x \in A$.

"$\Rightarrow$": Suppose $S \subseteq A$ is infinite. We want to prove that S has an accumulation point in A. Take $x_n \in S$ to be distinct for each $n \in \mathbb{N}$: this is possible since S is infinite. By assumption (x_n) has a subsequence (x_{n_k}) with $x_{n_k} \to x \in A$ as $k \to \infty$. Let's show that x is an accumulation point of S: if $\varepsilon > 0$, there is N s.t. $d(x_{n_k}, x) < \varepsilon$ for all $k \geq N$. Thus the points $x_{n_k} \in S$ are all in the disk $D(x, \varepsilon)$ and since there are infinitely many of them, x is an accumulation point of S.