Some solutions to HW5

<u>Problem 1</u>: Suppose for contradiction that $\mathbb{C}$ has an order relation $\leq$ that makes $\mathbb{C}$ into an ordered field. In that case the axioms imply that $-1 < 0$, but also that $x^2 \geq 0$ for all $x \in \mathbb{C}$. This is a contradiction since we would have

$$-1 < 0 \leq (i)^2 = -1 .$$

<u>Problem 2</u>: Let (M,d) be a finite metric space, and write $M = \{m_1, \dots, m_n\}$. We define $\varphi : M \to \mathbb{R}^n$ by $\varphi(m) = (d(m,m_1), \dots, d(m,m_n))$ for all $m \in M$.

<u>Claim</u>: $\|\varphi(m') - \varphi(m'')\|_\infty = d(m', m'')$ for all $m', m'' \in M$.

Let's prove it in two parts. Let $m', m'' \in M$.

"$\leq$": For any $i = 1, \dots, n$ we have two triangle inequalities:

$$\begin{cases} d(m_i, m'') + d(m'', m') \geq d(m_i, m') \\ d(m_i, m') + d(m', m'') \geq d(m_i, m'') \end{cases}$$

These together imply

$$-d(m', m'') \leq d(m_i, m') - d(m_i, m'') \leq d(m', m'')$$

$$\Rightarrow \quad |d(m_i, m') - d(m_i, m'')| \leq d(m', m'') \quad \text{for all } i = 1, \dots, n$$

so $\quad \|\varphi(m') - \varphi(m'')\|_\infty = \max_{i=1,\dots,n} |d(m_i, m') - d(m_i, m'')| \leq d(m', m'')$

"$\geq$": Note that $m' \in \{m_1, \dots, m_n\}$, so one of the $m_i = m'$. That means one of the coordinates in $\varphi(m') - \varphi(m'')$ is

$d(m', m') - d(m'', m') = -d(m'', m')$, so the sup-norm is at least $|-d(m'', m')| = d(m', m'')$. Thus

~~[illegible]~~ $\|\varphi(m') - \varphi(m'')\|_\infty \geq d(m', m'')$ $\square$.

Pg 63 #3) Orthogonal complement is the set of vectors $v=(x,y,z)$ that are orthogonal to both given vectors:

$$\langle v, (3,2,2)\rangle = 3x+2y+2z = 0$$
$$\langle v, (0,1,0)\rangle = y = 0$$

$\Rightarrow \begin{cases} 3x=-2z \\ y=0 \end{cases}$ so $(x,y,z) = t\left(\frac{-2}{3}, 0, 1\right)$ for some $t\in\mathbb{R}$.

Orth. compl. is $\left\{\left(\frac{-2}{3}t, 0, t\right) \mid t\in\mathbb{R}\right\}$.

Pg 70 #1) Sup norm: $\|f-g\|_\infty = \sup\{|f(x)-g(x)| \mid x\in[0,1]\} = \sup\{|1-x| \mid x\in[0,1]\} = 1$

L_2-norm: $\|f-g\|_2^2 = \int_0^1 (f(x)-g(x))^2 dx = \int_0^1 (1-x)^2 dx = \frac{1}{3}$

$$\|f-g\|_2 = \frac{1}{\sqrt{3}}.$$

#3) $\langle f,g\rangle = \int_0^1 1\cdot x\,dx = \frac{1}{2}$, $\langle f,f\rangle = \int_0^1 1\cdot 1\,dx = 1$, $\langle g,g\rangle = \int_0^1 x\cdot x\,dx = \frac{1}{3}$

Since $\underset{0.5}{\underbrace{\frac{1}{2}}} \leq \underset{0.577}{\underbrace{\sqrt{1}\cdot\sqrt{\frac{1}{3}}}}$, Cauchy inequality holds.

Pg 78 #3) No: e.g. $z=1+i$, $w=i$. $\Rightarrow zw=-1+i$

$Re(z)=1$, $Re(w)=0$, $Re(zw)=-1$.

#9) Since $|z_n| = n|z|^n$, it converges to 0 if $|z|<1$, and diverges if $|z|\geq 1$. (because then $|z_n|\to\infty$ as $n\to\infty$)