SOLUTIONS TO SOME OF HOMEWORK #1

1) a) $f(A) = \{1\}$, $f^{-1}(B) = S = \{1,0,-1\}$

b) $f(A) = \{x \in \mathbb{R} \mid x > 0\} = A$, $f^{-1}(B) = \{0\}$

c) $f(A) = \{1, 0, -1\}$, $f^{-1}(B) = \{x \in \mathbb{R} \mid x \leq 0\}$.

2) a) neither, b) both, c) neither

3) $f: S \to T$, $C_1, C_2 \subseteq T$, $D_1, D_2 \subseteq S$

a) Let's show equality: if $x \in S$,

$x \in f^{-1}(C_1 \cup C_2) \Leftrightarrow f(x) \in C_1 \cup C_2$

$\Leftrightarrow f(x) \in C_1$ or $f(x) \in C_2$

$\Leftrightarrow x \in f^{-1}(C_1)$ or $x \in f^{-1}(C_2)$

$\Leftrightarrow x \in f^{-1}(C_1) \cup f^{-1}(C_2)$

Thus $f^{-1}(C_1 \cup C_2) = f^{-1}(C_1) \cup f^{-1}(C_2)$.

b) First, let's show $f(D_1 \cup D_2) \subseteq f(D_1) \cup f(D_2)$: if $y \in T$, and $y \in f(D_1 \cup D_2)$, then $\exists x \in D_1 \cup D_2$ with $f(x) = y$.
This means that $x \in D_1$ or $x \in D_2$, so $f(x) = y \in f(D_1)$ or $y \in f(D_2)$.
Thus $y \in f(D_1) \cup f(D_2)$, and we get $f(D_1 \cup D_2) \subseteq f(D_1) \cup f(D_2)$.

Next, let's show $f(D_1) \cup f(D_2) \subseteq f(D_1 \cup D_2)$:
if $y \in f(D_1) \cup f(D_2)$, so $y \in f(D_1)$ or $y \in f(D_2)$. So, there is $x \in D_1$ or $x \in D_2$ with $y = f(x)$. This means, in both cases $x \in D_1 \cup D_2$, so $y \in f(D_1 \cup D_2)$, and
$f(D_1) \cup f(D_2) \subseteq f(D_1 \cup D_2)$.

c) If $x \in S$, then

$x \in f^{-1}(C_1 \cap C_2) \Leftrightarrow f(x) \in C_1 \cap C_2 \Leftrightarrow f(x) \in C_1$ and $f(x) \in C_2$

$\Leftrightarrow x \in f^{-1}(C_1)$ and $x \in f^{-1}(C_2) \Leftrightarrow x \in f^{-1}(C_1) \cap f^{-1}(C_2)$.

Thus $f^{-1}(C_1 \cap C_2) = f^{-1}(C_1) \cap f^{-1}(C_2)$.

d) Suppose $y \in f(D_1 \cap D_2)$. Then $\exists x \in D_1 \cap D_2$ such that $f(x) = y$.
Since $x \in D_1$ and $x \in D_2$, we see that $y \in f(D_1)$ and $y \in f(D_2)$, so $y \in f(D_1) \cap f(D_2)$.
Thus $f(D_1 \cap D_2) \subseteq f(D_1) \cap f(D_2)$.

5) Say $f: S \to T$ is a function. We want to show (a) $\Leftrightarrow$ (b) $\Leftrightarrow$ (c). Let's do this by showing implications (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (a).

(a) $\Rightarrow$ (b): Assume (a): f is injective. Then if $y \in T$ and $a, b \in f^{-1}(\{y\})$, then $f(a) = f(b) = y$. Since f is injective, this means $a = b$. This is true for any $a, b \in f^{-1}(\{y\})$ so $f^{-1}(\{y\})$ contains at most 1 element.

(b) $\Rightarrow$ (c): Assume (b). We want to show $f(D_1 \cap D_2) = f(D_1) \cap f(D_2)$ for all $D_1, D_2 \subseteq S$. By problem 3d), it is enough to show that $f(D_1) \cap f(D_2) \subseteq f(D_1 \cap D_2)$. Suppose $y \in f(D_1) \cap f(D_2)$. Then $y \in f(D_1)$ and $y \in f(D_2)$, so $\exists x_1 \in D_1$ and $x_2 \in D_2$ with $f(x_1) = f(x_2) = y$. Since $x_1, x_2 \in f^{-1}(\{y\})$, and $f^{-1}(\{y\})$ contains at most one element, we get $x_1 = x_2$. This means $x_1 \in D_1 \cap D_2$, and $y = f(x_1) \in f(D_1 \cap D_2)$. Thus $f(D_1) \cap f(D_2) \subseteq f(D_1 \cap D_2)$ and (c) is true.

(c) $\Rightarrow$ (a): Assume (c). We need to show that f is injective. Suppose $a, b \in S$ and they satisfy $f(a) = f(b)$. Define $D_1 = \{a\}$, $D_2 = \{b\} \subseteq S$ and by (c) we know

$$f(D_1 \cap D_2) = f(D_1) \cap f(D_2) = \{f(a)\} \cap \{f(b)\} = \{f(a)\}.$$

But if $a \neq b$, then $D_1 \cap D_2 = \emptyset$ so $f(D_1 \cap D_2) = \emptyset$, so we must have $a = b$. Thus f is injective.

12 (a) Write $\mathcal{A} = \{A_i \mid i \in I\}$, where I is "index set". Then $\mathcal{B} = \{S \setminus A_i \mid i \in I\}$.

- let's show $S \setminus \bigcup_{i \in I} A_i \subseteq \bigcap_{i \in I} (S \setminus A_i)$: if $x \in S \setminus \bigcup_{i \in I} A_i$, then $x \notin \bigcup_{i \in I} A_i$, so x doesn't belong to any A_i, so $x \in S \setminus A_i$ for all $i \in I$.
  Thus $x \in \bigcap_{i \in I} (S \setminus A_i)$.
- let's show $\bigcap_{i \in I} (S \setminus A_i) \subseteq S \setminus \left(\bigcup_{i \in I} A_i\right)$: if $x \in \bigcap_{i \in I} S \setminus A_i$, then $x \in S \setminus A_i$ for all i.
  This means that x is in none of A_i, so $x \notin \bigcup_{i \in I} A_i \Rightarrow x \in S \setminus \left(\bigcup_{i \in I} A_i\right)$.

Thus $S \setminus \bigcup_{i \in I} A_i = \bigcap_{i \in I} (S \setminus A_i)$.

Let's show $S \setminus \bigcap_{i \in I} A_i \subseteq \bigcup_{i \in I} (S \setminus A_i)$: if $x \in S \setminus \bigcap_{i \in I} A_i$, then $x \notin \bigcap_{i \in I} A_i$. Then $x \notin A_i$ for some $i \in I$, so $x \in S \setminus A_i$ for this i. Thus $x \in \bigcup_{i \in I} (S \setminus A_i)$.

Let's show $\bigcup_{i \in I} (S \setminus A_i) \subseteq S \setminus \bigcap_{i \in I} A_i$: if $x \in \bigcup_{i \in I} S \setminus A_i$, then there is some $i \in I$ with $x \in S \setminus A_i$. Then $x \notin A_i$ for this i, so $x \notin \bigcap_{i \in I} A_i$, $\Rightarrow x \in S \setminus \bigcap_{i \in I} A_i$.

Thus $S \setminus \bigcap_{i \in I} A_i = \bigcup_{i \in I} (S \setminus A_i)$.

15) a) We want to prove: $f: S \to T$ injective $\Leftrightarrow \exists g: T \to S$ s.t. $g \circ f = id_S$.

"$\Leftarrow$": if such $g: T \to S$ exists, then for all $a, b \in S$ we have

$$f(a) = f(b) \Rightarrow g(f(a)) = g(f(b)) \Rightarrow (g \circ f)(a) = (g \circ f)(b) \Rightarrow a = b \text{, as } g \circ f = id.$$

Thus f injective.

"$\Rightarrow$": Suppose $f: S \to T$ injective. Pick an arbitrary $s_0 \in S$ (we assume $S \neq \emptyset$), and define $g: T \to S$ as follows: for $t \in T$, if $t \notin f(S)$, set $g(t) = s_0$.

If $t \in f(S)$, then $f^{-1}(\{t\})$ has exactly one element s; set $g(t) = s$.

Then for all $s \in T$, $g(f(s)) = s$ by definition.

b) We want to prove $f: S \to T$ surjective $\Leftrightarrow \exists h: T \to S$ s.t. $f \circ h = id_T$.

"$\Leftarrow$": if such $h: T \to S$ exists, then for all t we have $f(h(t)) = t$ so $t \in f(S)$.

Thus $f(S) = T$ and f surjective.

"$\Rightarrow$": Suppose $f: S \to T$ is surjective. We want to define $g: T \to S$.

If $t \in T$, pick an arbitrary $s_t \in S$ in $f^{-1}(\{t\}) \neq \emptyset$, and set $g(t) = s_t$.

Then for all $t \in T$ we have $f(g(t)) = t$, so $f \circ g = id_T$

(Apparently I changed from h to g)

c) We want to prove: $f: S \to T$ bijective $\Leftrightarrow \exists g: T \to S$ s.t. $f \circ g = id_T$ and $g \circ f = id_S$.

"$\Leftarrow$" if $g: T \to S$ is both left and right inverse, then by (a) and (b) f is both injective and surjective. Thus f is bijective.

"$\Rightarrow$" if f is bijective, then by (a) and (b) there are $g, h: T \to S$ with $g \circ f = id_S$ and $f \circ h = id_T$. Then

$$g = g \circ id = g \circ (f \circ h) = (g \circ f) \circ h = id \circ h = h.$$

Thus $g = h$ is both left and right inverse of f.

16) By 15 c) we only need to show that $f^{-1} \circ g^{-1}$ is inverse function of $g \circ f$:

$$f^{-1} \circ g^{-1} \circ g \circ f = f^{-1} \circ id \circ f = f^{-1} \circ f = id, \quad g \circ f \circ f^{-1} \circ g^{-1} = g \circ id \circ g^{-1} = g \circ g^{-1} = id.$$

Thus $g \circ f$ has an inverse and it is bijective.