Math 240 Fall 2011
Quiz 5 Name: ______________________________
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Show all work clearly and in order, and box your final answers. You have 10 minutes to take this quiz.

1. (*15 points*) Use Stokes' theorem to find the surface integral

$$\iint_S (\nabla \times \bar{F}) \cdot \bar{n}\, \mathrm{d}S,$$

where $\bar{F}(x, y, z) = y\,\bar{\imath} - x\,\bar{\jmath} + (xy - z^2)\,\bar{k}$ and S is the upper hemisphere

$$x^2 + y^2 + z^2 = 1 \quad \text{with} \quad z \geq 0.$$

(Note: here you should use Stokes' theorem "backwards" compared to how you usually use it.)

Solution: By Stokes' theorem,

$$I := \iint_S (\nabla \times \bar{F}) \cdot \bar{n}\, \mathrm{d}S = \oint_C \bar{F} \cdot \mathrm{d}\bar{r},$$

where C is the boundary of the hemisphere. In other words, C is the circle $x^2 + y^2 = 1$ in the xy-plane.
We can parametrize C by

$$\begin{cases} x = \cos(\theta) \\ y = \sin(\theta) \\ z = 0 \end{cases}, \quad \text{which gives} \quad \begin{cases} \mathrm{d}x = -\sin(\theta)\mathrm{d}\theta \\ \mathrm{d}y = \cos(\theta)\mathrm{d}\theta \\ \mathrm{d}z = 0. \end{cases}$$

Thus the asked integral is

$$\begin{aligned} I &= \oint_C y\, \mathrm{d}x - x\, \mathrm{d}y + (xy - z^2)\, \mathrm{d}z = \\ &= \int_{\theta=0}^{2\pi} (-\sin(\theta)\sin(\theta) - \cos(\theta)\cos(\theta))\mathrm{d}\theta = \\ &= -\int_{\theta=0}^{2\pi} (\sin^2(\theta) + \cos^2(\theta))\mathrm{d}\theta = -2\pi. \end{aligned}$$