A New Fibonacci Identity

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Tiling a Board

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A tiling of a board of length n and height 1 consists of a non-overlapping placement of squares (1×1) and dominoes (2×1) which completely cover the board.

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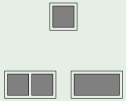

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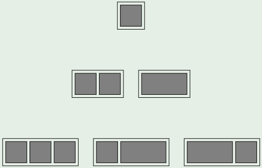

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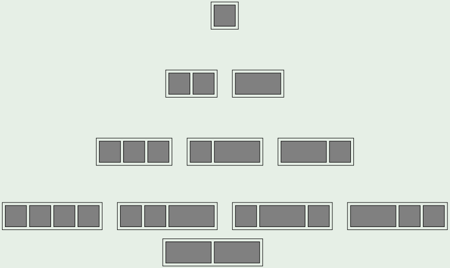

Counting Board Tilings

n	1	2	3	4	5	6	7	8
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Theorem

There are F_n ways to tile a board of length n with squares and dominoes, where $F_0 = F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$.

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Theorem

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Example

When $n = 4$, we have $F_{2n} = F_8 = 34$ and
$(F_n)^2 + (F_{n-1})^2 = (F_4)^2 + (F_3)^2 = 5^2 + 3^2 = 34$.

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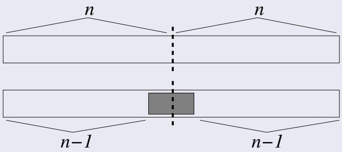

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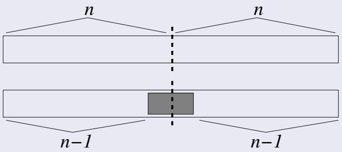

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Theorem (Lonoff, Ostroff)

$$\sum_{k=1}^{n} F_{2k-4} 2^{n-k} = F_{2n-1}, \textit{ for all } n \geq 1.$$

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Answer 1: F_{2n-1}

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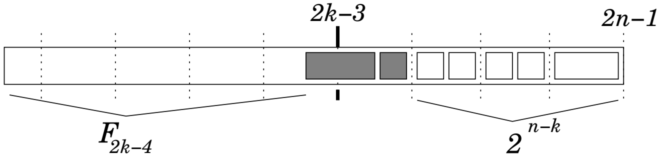

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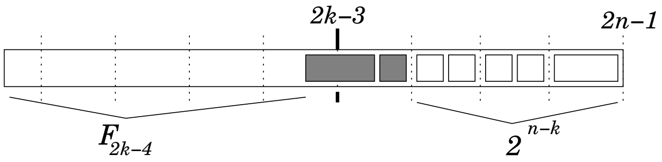
Answer 2: $\sum_{k=1}^{n} F_{2k-4} 2^{n-k}$



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$$\sum_{k=1}^{n} F_{2k-4} 2^{n-k} = F_{2n-1}, \text{ for all } n \geq 1.$$

This naturally generalizes to

$$F_{mn+r} = F_r F_m^n + \sum_{k=1}^{n} F_{mk-m+r-1} F_{m-1} F_m^{n-k}$$

Acknowledgements

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Thanks for your attention!