Symmetric Permutations Avoiding Two Patterns

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Permutations and Patterns

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A permutation of $1, 2, \ldots, n$ is a listing of $1, 2, \ldots, n$ in some order. S_n is the set of all permutations of length n.

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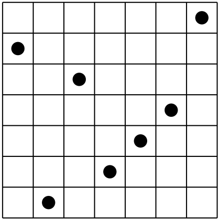

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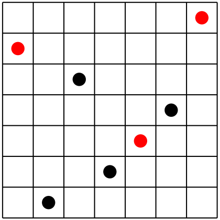

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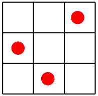

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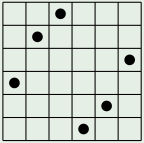

$356124 \in S_6^{rc}$

Counting Symmetric Permutations

Theorem

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A signed permutation is a permutation written in one-line notation in which each entry may or may not have a bar above it. B_n is the set of signed permutations of length n.

Clearly $|B_n| = 2^n n!$. Thus, it suffices to find bijections from B_n to S_{2n}^{rc} and S_{2n+1}^{rc}.

Counting Symmetric Permutations

$s : B_n \mapsto S^{rc}_{2n}$

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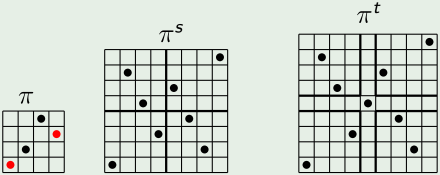
Example

$\pi = \bar{1}24\bar{3}$



Symmetric Pattern-avoiding Permutations

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Let R be a set of permutations, possibly of different lengths.
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Theorem (Egge)

$|S_{2n+1}^{rc}(123)| = C_n = \frac{1}{n+1}\binom{2n}{n}$

Theorem (Egge)

$|S_{2n}^{rc}(132)| = 2^n$

Theorem (Egge)

$|S_{2n}^{rc}(123)| = \binom{2n}{n}$

Theorem (Egge)

$|S_n^{rc}(132, 231)| = 2$

$S_n^{rc}(123, 4231)$

Theorem

$|S_{2n}^{rc}(123, 4231)| =$

Suppose $\pi \in B_n$. Under what conditions is $\pi^s \in S_{2n}^{rc}(123, 4231)$?

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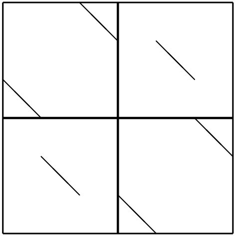

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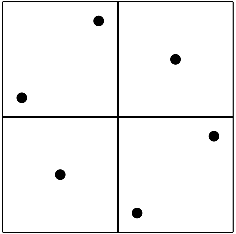

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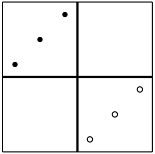

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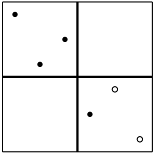

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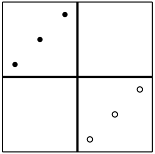

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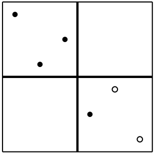

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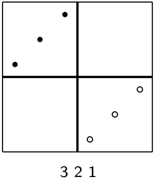

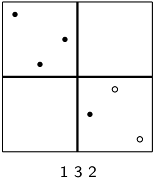

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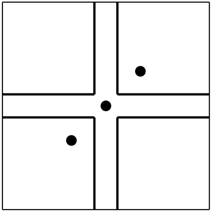

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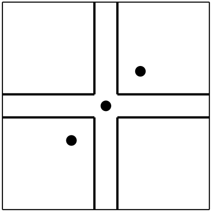

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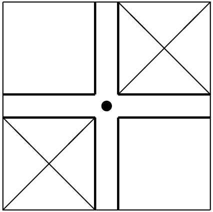

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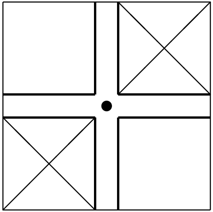

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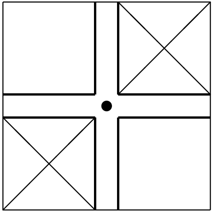

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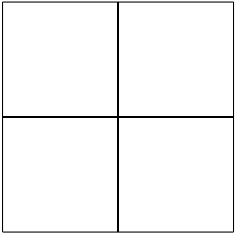

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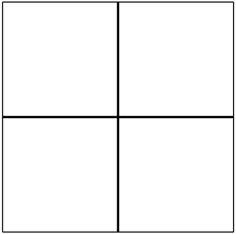

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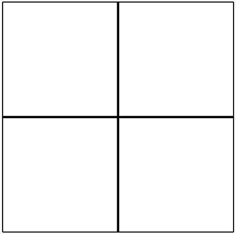

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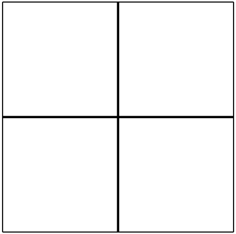

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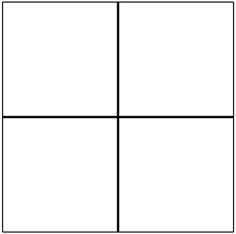

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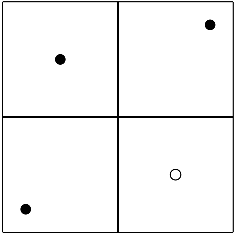

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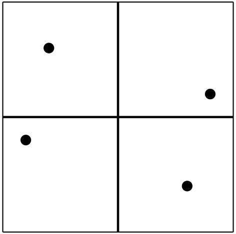

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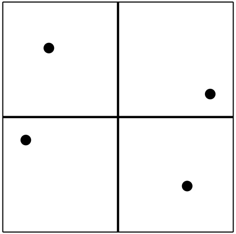

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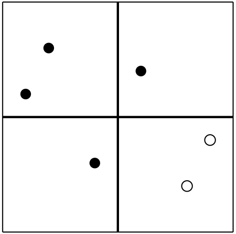

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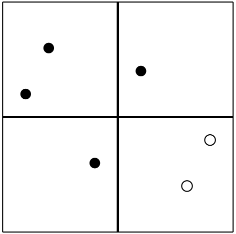

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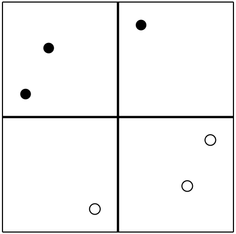

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1. Non-bars avoid 321 and 3142
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$$F_{2n-1} + \sum_{k=1}^{n} F_{2(k-1)-1} 2^{n-k}$$

Our Results

Theorems (Lonoff, Ostroff)

σ	τ	$\lvert S_{2n}^{rc}(\sigma,\tau)\rvert$	$\lvert S_{2n+1}^{rc}(\sigma,\tau)\rvert$
123	2413	F_{2n+1}	F_{2n-1}
123	2431	$F_{n+3}+1$	$F_{n+2}-1$
123	3412	$2^{n+1}-(n+1)$	1
123	4231	n^2+1	$\binom{n}{2}+1$
123	4312	6	1
123	1432	*A166963*	*A116716*
132	1234	T_n	T_n
132	2341	$F_{n+1}+1$	F_n+1
132	3412	$n+1$	$n+1$
132	4231	$n+1$	$n+1$
132	4321	$n+1$	$n+1$
132	3421	4	3

(Where $F_0 = F_1 = 1$,
$F_n = F_{n-1} + F_{n-2}$
for $n \geq 2$, *and*
$T_0 = 1$, $T_1 = 2$, $T_2 = 3$,
$T_n = T_{n-1} + T_{n-2} + T_{n-3}$
for $n \geq 3$.*)*

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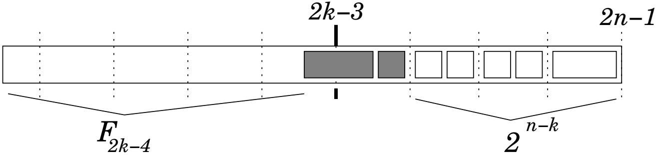

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This naturally generalizes to

$$F_{mn+r} = F_r F_m^n + \sum_{k=1}^{n} F_{mk-m+r-1} F_{m-1} F_m^{n-k}$$

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