

## Two Color Ramsey Numbers

- Example: Amongst any six people, there are three who mutually know one another or three who mutually do not know one another.
  - In graph terms, if we color each edge of  $K_6$  red or blue, we will get a monochromatic  $K_3$  (triangle) as a subgraph.
  - Let  $R(n, m)$  be the smallest  $L$  such that if we color each edge of  $K_L$  red or blue, we either get a red  $K_n$  or a blue  $K_m$ .
  - Shorthand  $R(n) = R(n, n)$ . So the example says that  $R(3) = 6$ .
- $R(n, m)$  always exists
  - $R(n, 2) = n$  for all  $n$ .
  - $R(n, m) \leq R(n-1, m) + R(n, m-1)$ .
  - These show, inductively, that  $R(n, m) \leq \binom{n+m-2}{m-1}$ .
- Some examples of Ramsey numbers.
  - $R(4, 3) = 9$ .
  - $R(4, 4) = 18$ .

## Generalizations

- $R(n, m, l)$  is smallest  $L$  so that coloring the edges of  $K_L$  red, blue, or green yields a monochromatic  $K_n$ ,  $K_m$ , or  $K_l$ , in the respective color.
  - The only non-trivial number known is  $R(3, 3, 3) = 17$ .
  - Upper bound (easy induction) is  $R(3, \dots, 3) \leq 3c!$ .
  - Best known upper bound (trickier induction) is  $\lfloor ec! \rfloor + 1$ .

## Hypergraph Ramsey Theory

- $K_m^t$  denotes the complete hypergraph on  $m$  vertices where edges are the  $t$ -element subsets of the vertices.
  - Let  $R_t(n, m)$  be the minimum  $L$  such that coloring the edges of  $K_L^t$  red or blue yields a red  $K_n^t$  or a blue  $K_m^t$ .
  - These numbers exist, but are much harder to bound.

## Infinite Ramsey Theory

- Any finite coloring of edges of  $K_{\mathbb{N}}$  yields a monochromatic  $K_{\mathbb{N}}$ . (Corollary: Every sequence of real numbers contains a monotone subsequence)
- If we allow countably many colors for edge coloring  $K_{\mathbb{N}}$ , then no monochromatic triangle (duh, since there are only countably many edges)

- If we allow countably many colors for edge coloring  $K_c$ , then we can avoid monochromatic triangles!

### Other Related Theorems

- Van Der Waerden's Theorem: Any  $r$ -coloring of  $\mathbb{N}$  contains monochromatic arithmetic progressions of arbitrary length.

- HUGE upper bounds. Current record (Gowers, 2001):

$$W(r, k) \leq 2^{2^{r \cdot 2^{k+9}}}$$

- \$ 1000 Prize if you can prove

$$W(2, k) \leq 2^{k^2}.$$

- This would still be a pretty bad bound, since  $W(2, 3) = 9$ ,  $W(2, 4) = 35$ ,  $W(2, 5) = 178$ .

- Schur's Theorem: Given any finite coloring of  $\mathbb{N}$  there exists a monochromatic solution to  $x + y = z$ .

- Let  $S(c)$  denote the smallest integer such that a  $c$ -coloring of  $[S(c)]$  yields a monochromatic solution. Then  $S(c) \leq R(3, \dots, 3)$ .

- Rado's Theorem (generalizes Schur): Given a finite coloring of  $\mathbb{N}$ , there exists a monochromatic solution to

$$c_1x_1 + \dots + c_mx_m = 0$$

if and only if there exists a subset of the  $c_i$  whose sum is zero.

### The Happy Ending Problem

- (Erdős-Szekeres 1935) Given any five points in the plane in general position (no three on a line), four of the points determine a convex quadrilateral.
- More generally, for any  $n$ , there exists  $f(n)$  such that among any  $f(n)$  points in the plane there are  $n$  which determine a convex  $n$ -gon.
- Proof:  $f(n) \leq R_3(n, n)$ . Also  $f(n) \leq R_4(5, n)$ .

### Other Applications

- Number theory
- Set theory
- Geometry
- Topology
- Ergodic Theory
- Mathematical logic
- Ultrafilters