Problem: Show that when n is a positive integer,

$$\sum_{k\geq 0}\binom{n}{k}\binom{2k}{k}=\sum_{k\geq 0}\binom{n}{2k}\binom{2k}{k}3^{n-2k}.$$

Solution I by CMC 328, Carleton College, Northfield, MN. Consider a country with n states, in which each state has a senior senator and a junior senator (distinguishable). Each senator is Democratic (D), Republican (R), or Independent (I). States are either *independent* (both senators Independent) or *partisan* (neither senator Independent); no state has exactly one Independent senator. A *balanced senate* is an assignment of parties to the $2n$ senators so that the number of Republican senators equals the number of Democratic senators. We show that both sides of the equation count the balanced senates.

Left side: There are $\binom{n}{k}$ ways to choose k states to be partisan. There are $\binom{2k}{k}$ ways to assign their senators to parties. Thus the number of balanced senates is $\displaystyle\sum_{k\geq 0}\binom{n}{k}\binom{2k}{k}$.

Right side: The number of states with two Republicans equals the number with two Democrats; let this number be k. There are $\binom{n}{2k}$ ways to choose these states, and there are $\binom{2k}{k}$ ways to assign their senators to parties. Each remaining state is R/D, D/R, or I/I; these can be assigned in 3^{n-2k} ways. Counted this way, the number of balanced senates is $\displaystyle\sum_{k\geq 0}\binom{n}{2k}\binom{2k}{k}3^{n-2k}$.

[...]

CMC 328 also observed that for senates with $2m$ more Democrats than Republicans, the identity generalizes to

$$\sum_{k\geq 0}\binom{n}{k}\binom{2k}{k+m}=\sum_{k\geq 0}\binom{n}{2k+m}\binom{2k+m}{k}3^{n-(2k+m)}.$$

To allow p independent parties (with each independent state having two senators of the same independent party), multiply the summand on the left by p^{n-k}, and change 3 to $2+p$ on the right.