Problem: Given integers $i_1, \ldots, i_k$, and n, prove that

$$\sum_{j=0}^{n} \binom{n}{j} \prod_{r=1}^{k} \binom{j}{i_r}$$

is divisible by 2^{n-i}, where $i = i_1 + i_2 + \ldots + i_k$.

Solution: Suppose there is a party with n attendees. At some point in the evening, the host decides to play some party games and asks everyone who is interested in playing to put on a hat. Let there be k party games, where the rth game requires exactly i_r participants. These participants are chosen, with replacement, from the people with hats. In how many ways can we select people to wear hats and from those people choose people to play the k different games?

Answer 1: Condition on the number of attendees, say j, who put on hats. First, choose the j people to wear hats in $\binom{n}{j}$ ways. Then for the rth game, choose i_r people from the j people with hats, which can be done in $\binom{j}{i_r}$ ways. Since each game is independent, the number of ways we can make selections for all k games is $\prod_{r=1}^{k} \binom{j}{i_r}$. Summing over all j gives

$$\sum_{j=0}^{n} \binom{n}{j} \prod_{r=1}^{k} \binom{j}{i_r}.$$

Answer 2: Condition on the number of attendees, say w, who actually play in at least one game. Note that w is at most $i = i_1 + i_2 + \ldots + i_k$, which is achieved if each person plays in at most one game. Also, w is at least $w_{\min} = \max\{i_1, \ldots, i_k\}$ because there must be at least as many players as the largest game requires. The w attendees chosen automatically wear hats, and we may also choose additional attendees to wear hats but not to play in any games. For each non-player (there are $n - w$ of them), we can either give them a hat or not. Thus, the answer is

$$\sum_{w=w_{\min}}^{i} (\# \text{ ways for } w \text{ players to play in } k \text{ games}) \cdot 2^{n-w},$$

which clearly has 2^{n-i} as a factor.