Bijective Proofs: You say you want an involution?

David "D\$" Lonoff and Daniel "DMac" McDonald

Carleton College

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Definition

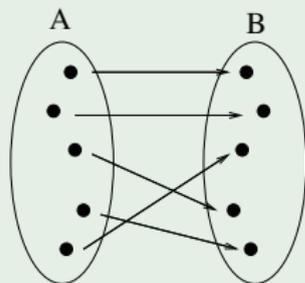A bijection is a one-to-one and onto mapping.

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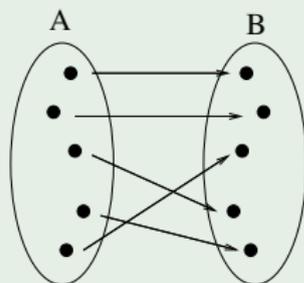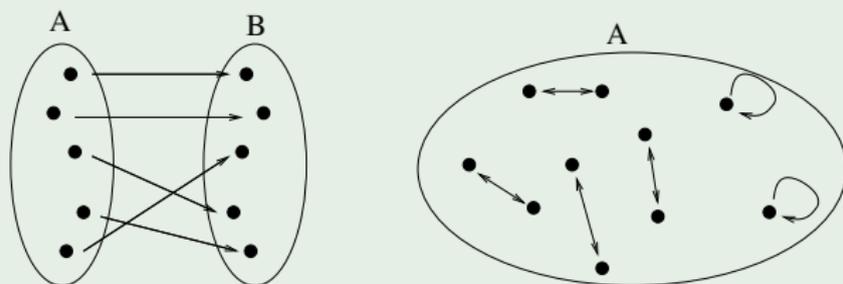**Definition**

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$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

($\binom{n}{k} = \frac{n!}{k!(n-k)!}$ is the number of unordered subsets of size k from a set of size n)

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Example

Are there an even or odd number of people in the room right now?

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A partition λ of n with parts $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_s$ is either written as such or in the form $\lambda_{n_1}^{m_1} \lambda_{n_2}^{m_2} \ldots \lambda_{n_k}^{m_k}$, where each λ_{n_i} is a distinct part in λ with $\lambda_{n_1} > \lambda_{n_2} > \ldots > \lambda_{n_k}$, and there are m_i copies of λ_{n_i}.

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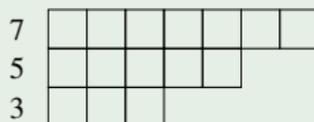
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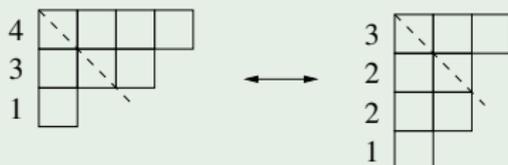
Euler's Theorem

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Let $O(n)$ denote the set of partitions of n into odd parts, and let $D(n)$ denote the set of partitions of n into distinct parts. Then for all n, $|O(n)| = |D(n)|$.

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Example

$D(6) : \{6, 5 + 1, 4 + 2, 3 + 2 + 1\}$
$O(6) : \{5 + 1, 3 + 3, 3 + 1 + 1 + 1, 1 + 1 + 1 + 1 + 1 + 1\}$

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We give a bijection $\phi : D(n) \to O(n)$ devised by Glaisher (1883). For $\lambda = \lambda_1 > \lambda_2 > \ldots > \lambda_s \in D(n)$, let $\lambda_i = 2^{p_i}\mu_i$, where μ_i is odd.

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Theorem

Let $Q(n)$ be the set of partitions of n into distinct parts where the smallest part is odd. Then $|Q(n)|$ is odd if and only if n is a perfect square.

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- $Q(6)$: $\{5+1, 3+2+1\}$
- $Q(7)$: $\{7, 6+1, 4+3, 4+2+1\}$
- $Q(9)$: $\{9, 8+1, 6+3, 6+2+1, 5+3+1\}$
- $Q(10)$: $\{9+1, 7+3, 7+2+1, 6+3+1, 5+4+1, 4+3+2+1\}$

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Idea of the proof: Define an involution $\kappa : Q(n) \rightarrow Q(n)$ which has exactly one fixed point if and only if n is a perfect square (and has no fixed points if and only if n is not a perfect square).

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Now, we construct $\kappa(\lambda)$ by performing one of the following three actions, depending on the case.

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3. If $t_\alpha = e_1 = \infty$, then do nothing.
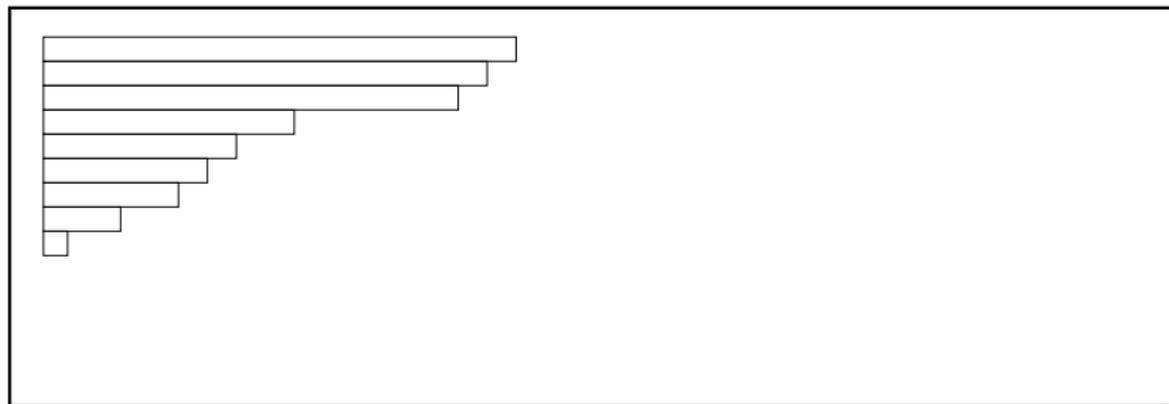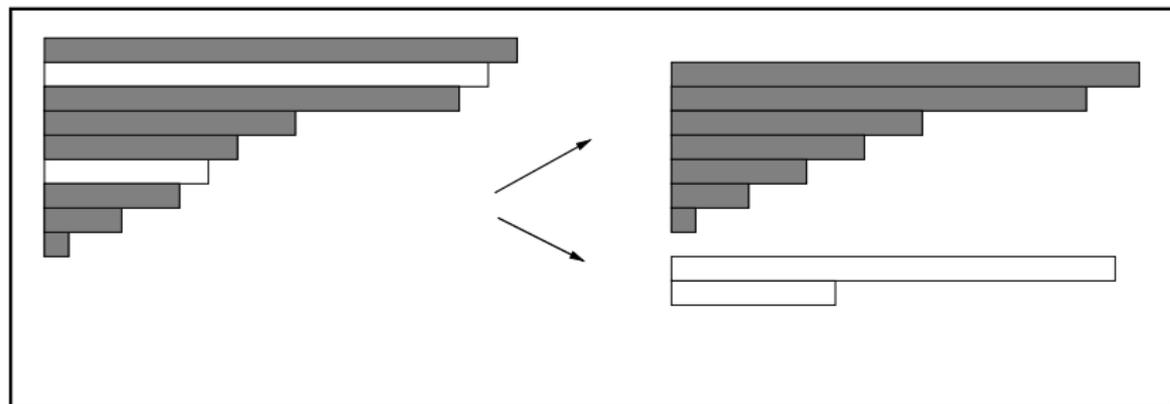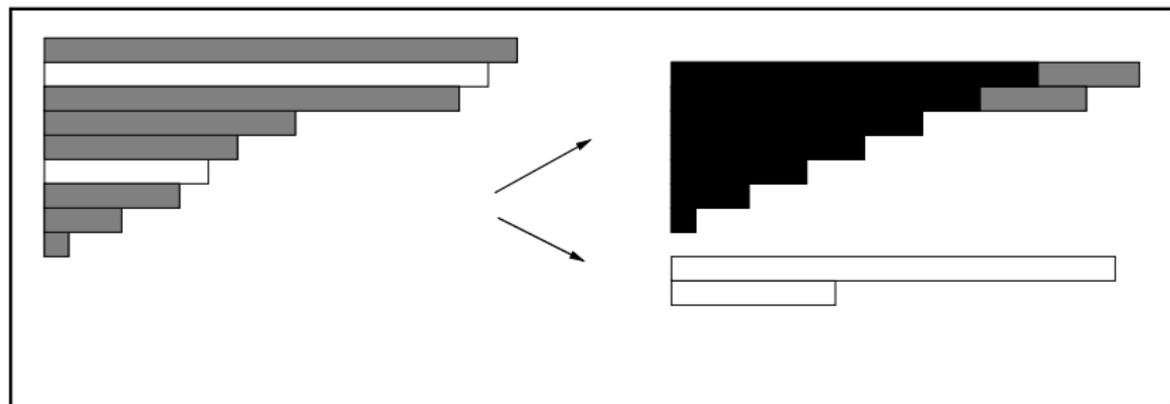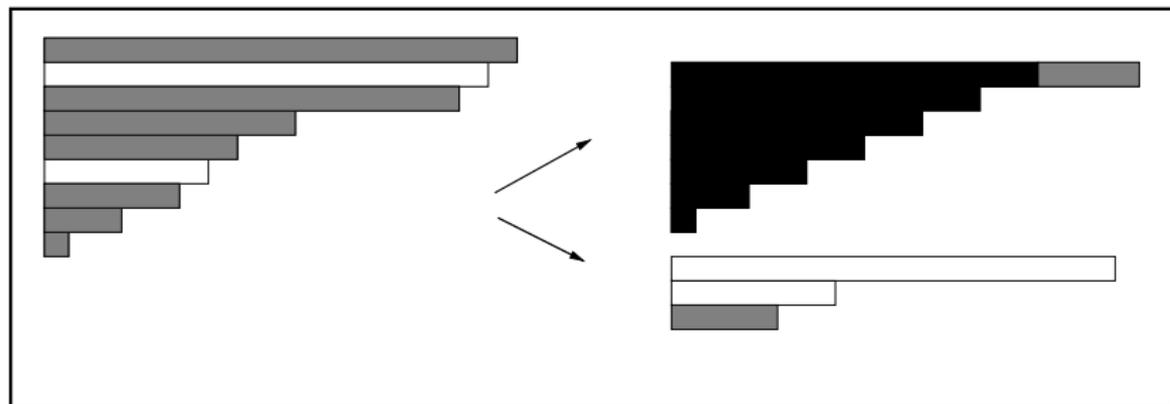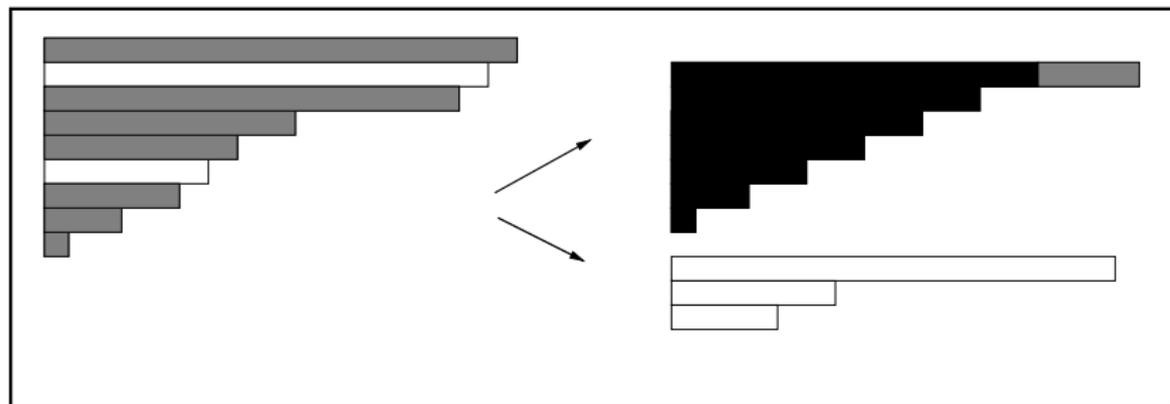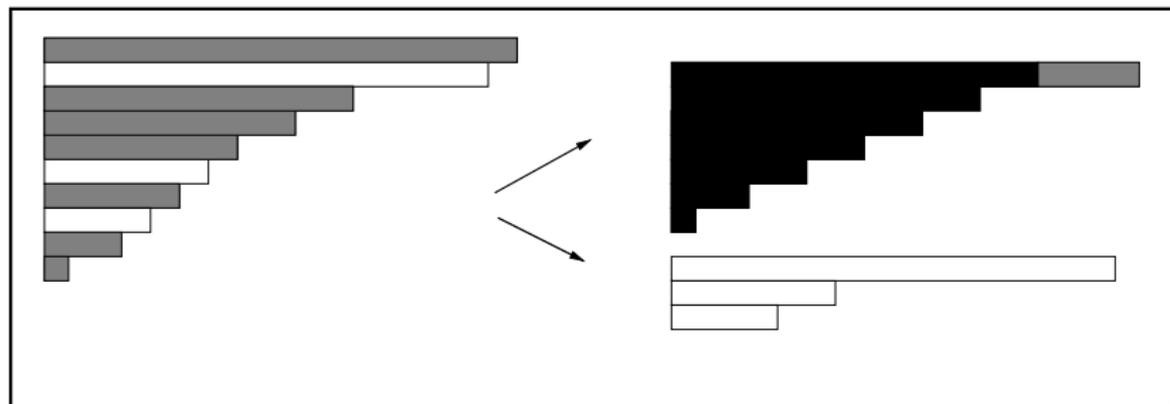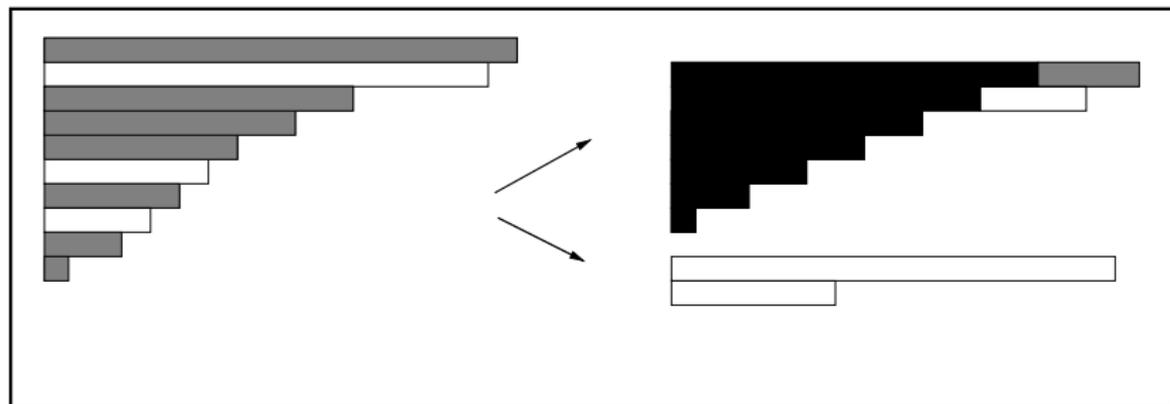
Kolberg's Theorem

The partition function $p(n)$ takes infinitely many even and odd values.

Let $SC(n)$ denote the set of self-conjugate partitions of n.

Proof of Kolberg's Theorem

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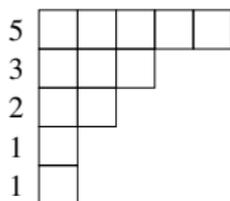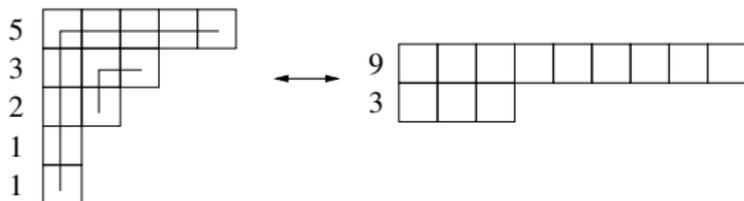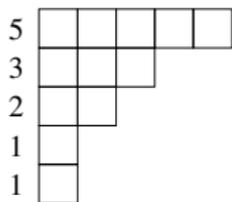Then $p(n)$ has the same parity as $|SC(n)|$.
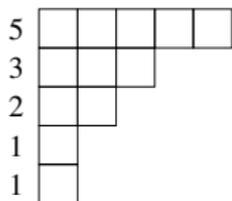
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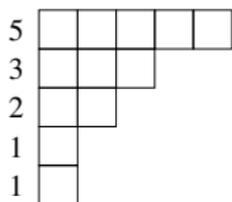
Let $DO(n) = D(n) \cap O(n)$ be the set of partitions of n into distinct odd parts.

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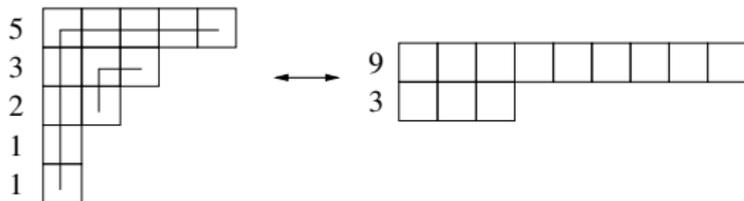Then $|SC(n)| = |DO(n)|$.

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Let $DO(n) = D(n) \cap O(n)$ be the set of partitions of n into
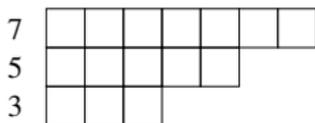distinct odd parts.
Then $|SC(n)| = |DO(n)|$.

Thus $p(n)$ has the same parity as $|DO(n)|$.

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For any $\lambda \in DO(n)$ denote the parts of λ by $\lambda_1 > \lambda_2 > \ldots > \lambda_s$.
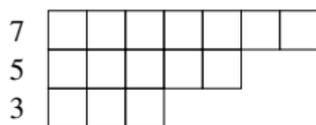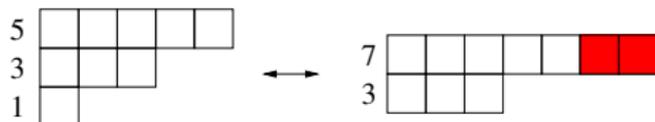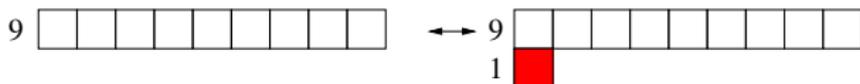
Proof of Kolberg's Theorem

For any $\lambda \in DO(n)$ denote the parts of λ by $\lambda_1 > \lambda_2 > \ldots > \lambda_s$.
Let $DO_1(n) = \{\lambda \in DO(n) | \lambda_1 - \lambda_2 = 2, \lambda_s > 1\}$.

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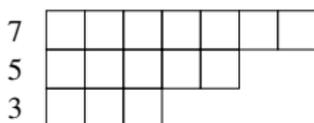Let $DO_1(n) = \{\lambda \in DO(n) | \lambda_1 - \lambda_2 = 2, \lambda_s > 1\}$.

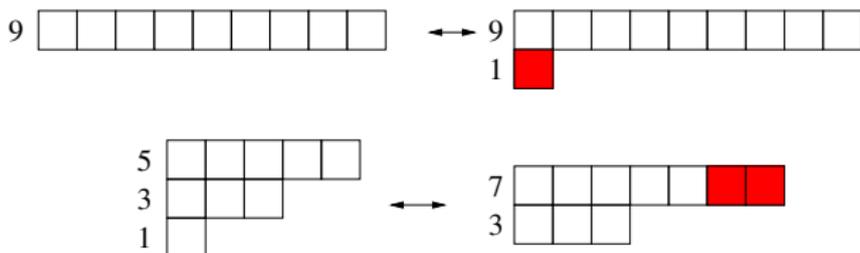Then $|DO(n)| = |DO(n+1) \backslash DO_1(n+1)|$.

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Then $|DO(n)| = |DO(n+1) \backslash DO_1(n+1)|$.

Thus $|DO(n)|$ differs in parity from $|DO(n+1)|$ precisely when $|DO_1(n+1)|$ is odd.

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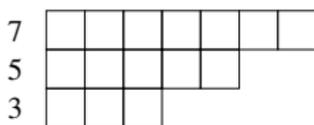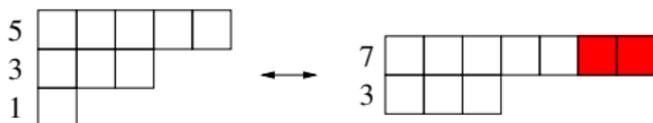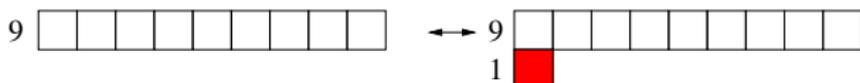Let $DO_1(n) = \{\lambda \in DO(n) | \lambda_1 - \lambda_2 = 2, \lambda_s > 1\}$.

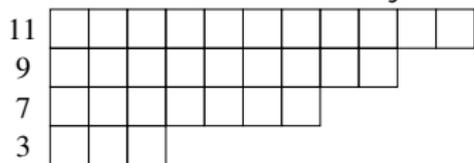Then $|DO(n)| = |DO(n+1) \backslash DO_1(n+1)|$.

Thus $|DO(n)|$ differs in parity from $|DO(n+1)|$ precisely when $|DO_1(n+1)|$ is odd.
Thus to prove the theorem we can show that $|DO_1(n)|$ is odd for infinitely many n.

Let $DO_j(n) =$
$\{\lambda \in DO(n) | \lambda_1 - \lambda_2 = \lambda_2 - \lambda_3 = \ldots = \lambda_j - \lambda_{j+1} = 2, \lambda_s > 1\}$.

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Then for all positive integers j,
$|DO_j(n)| = |DO_j(n + 2j + 2) \setminus DO_{j+1}(n + 2j + 2)|$.

$$\phi_2 \colon DO_2(24) \longrightarrow DO_2(30) \setminus DO_3(30)$$
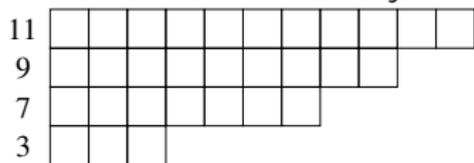
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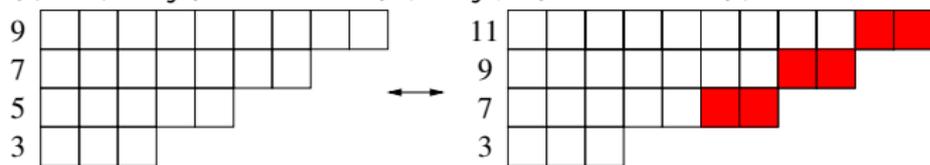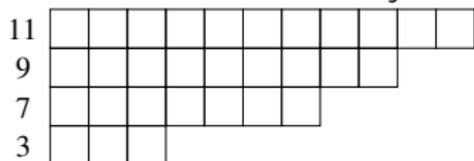

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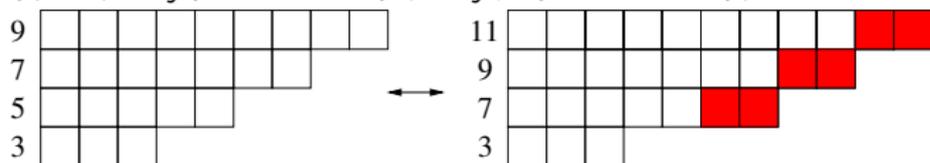
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when $|DO_{j+1}(n + 2j + 2)|$ is odd.

Given $k \geq 1$, we construct $n > k$ such that $|DO_1(n)|$ is odd.

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$n_1 \geq k^2 - 1 - (2(k-2) + 2(k-3) + \ldots + 4) =$
$k^2 - 1 - 2(2 + 3 + \ldots + k - 2) = k^2 - 1 - k(k-3) = 3k - 1 > k.$

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